Hahn-Banach sans Zorn

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Abstract. We prove the Hahn-Banach Theorem using the ultrafilter lemma.

1 All things ultra

Definition 1.1. A **filter** $\mathcal{F}$ on a set X is a nonempty subset $\mathcal{F} \subseteq \mathscr{P}(X)$ such that

(i) (non-trivial) $\emptyset \notin \mathcal{F}$;

(ii) (upwards closed) if $x \in \mathcal{F}$ and $x \subseteq y$ then $y \in \mathcal{F}$;

(iii) ($\aleph_0$-complete) if $x, y \in \mathcal{F}$ then $x \cap y \in \mathcal{F}$. $\dashv$

Definition 1.2. An **ultrafilter** $\mathcal{U}$ on a set X is a filter on X such that

(i) (ultra property) for any $x \in \mathscr{P}(X)$, either $x \in \mathcal{U}$ or $\neg x \in \mathcal{U}$. $\dashv$

An ultrafilter can equivalently be characterised as a filter which is maximal with respect to inclusion. This reminds us of Zorn's Lemma and indeed, in ZFC we can always extend a filter to an ultrafilter.

Lemma 1.3 (Ultrafilter Lemma; ZFC). *Every filter can be extended to an ultrafilter.* $\dashv$

It turns out that the ultrafilter lemma, henceforth UL, lies somewhere strictly between ZF and ZFC.

Proposition 1.4. *ZF + UL does not imply ZFC and ZF does not imply ZF + UL.* $\dashv$

Definition 1.5. Let $\mathcal{U}$ be an ultrafilter on X. Then the **ultrapower** associated to $\mathcal{U}$ is the structure $\mathrm{Ult}(V, \mathcal{U}) = (V^X / \sim_{\mathcal{U}}, E)$, where

$$V^X := \{f \mid \text{"}f \text{ is a function with } \operatorname{dom} f = X\text{"}\}$$

and $f \sim_{\mathcal{U}} g$ iff $\{x \in X \mid f(x) = g(x)\} \in \mathcal{U}$. We also define $f \in_{\mathcal{U}} g$ iff $\{x \in X \mid f(x) \in g(x)\} \in \mathcal{U}$. The **ultrapower embedding** is the function $j : V \to \mathrm{Ult}(V, \mathcal{U})$ given as $j(x) := [c_x]$ with $c_x : X \to \{x\}$ being the constant function.[1] $\dashv$

Proposition 1.6. *Let $\mathcal{U}$ be an ultrafilter on X with $j : (V, \in) \to (\mathrm{Ult}(V, \mathcal{U}), \in_{\mathcal{U}})$ the ultrapower embedding. Then*

(i) whenever $x_1, \ldots, x_n$ are sets and $\varphi(v_1, \ldots, v_n)$ is some formula[2]

$$(V, \in) \models \varphi[x_1, \ldots, x_n] \quad \text{iff} \quad (\mathrm{Ult}(V, \mathcal{U}), \in_{\mathcal{U}}) \models \varphi[j(x_1), \ldots, j(x_n)];$$

(ii) to every $y \in_{\mathcal{U}} j(\mathbb{R})$ there's a unique $x \in \mathbb{R}$ such that $|j(x) - y|$ is infinitesimal[3] $\dashv$

From the above (ii) define $\mathsf{st} : (j(\mathbb{R}), \in_{\mathcal{U}}) \to (\mathbb{R}, \in)$ as taking $y \in_{\mathcal{U}} j(\mathbb{R})$ to the above-mentioned unique $x \in \mathbb{R}$, called the *standard part* of y.

Proposition 1.7. *$\mathsf{st} : (j(\mathbb{R}), \in_{\mathcal{U}}) \to (\mathbb{R}, \in)$ is a homomorphism.* $\dashv$

2 Hahn-Banach

Lemma 2.1 (Finite extension lemma; ZF). *Let X be a normed space, $p : X \to \mathbb{R}$ a seminorm, $Z \subseteq X$ a subspace and $f \in Z^*$ a functional which satisfies $|f| \leqslant p$. Then for every $x_0 \in X$ we can construct a functional $\bar{f} \in \mathrm{span}(Z \cup \{x_0\})^*$ which extends f and which still satisfies $\left|\bar{f}\right| \leqslant p$.*

Proof. Assume we did have such an extension $\bar{f}$. Then by linearity and the triangle equality we get that for any $\lambda \in \mathbb{F}$ and $z \in Z$,

$$f(z) + \lambda \bar{f}(x_0) = \bar{f}(z + \lambda x_0) \leqslant p(z + \lambda x_0).$$

If $\lambda > 0$ this means that

$$\bar{f}(x_0) \leqslant \inf\{p(z/\lambda + x_0) - f(z/\lambda) \mid z \in Z\}$$

[1] Here V is the class of all sets. Strictly speaking $\mathrm{Ult}(V, \mathcal{U})$ doesn't make sense, as the equivalence classes might be proper classes. This can be fixed by using a trick due to Scott, which is picking representatives of minimal rank. We suppress these details here.

[2] Here "formula" means a first-order formula in the language of set theory $\{\in\}$. Also, "$\mathcal{M} \models \varphi$" is the *satisfaction predicate* and is to be understood as φ being true inside $\mathcal{M}$; i.e. with quantifiers relative to $\mathcal{M}$.

[3] Here $z \in_{\mathcal{U}} j(\mathbb{R})$ is *infinitesimal* means that $z > 0$ but $z < x$ for every $x \in \mathbb{R}$, which can be shown to exist.

and if $\lambda < 0$, then substituting λ for $-\lambda$ we get that

$$\bar{f}(x_0) \geqslant \sup\{f(z/\lambda) - p(z/\lambda - x_0) \mid z \in Z\}.$$

It remains to see if we can find anything that satisfies these two conditions. But in general, for any $z_1, z_2, y \in X$ it holds that

$$\begin{aligned} f(z_1) + f(z_2) &= f(z_1 + z_2) \\ &\leqslant p(z_1 + z_2) \\ &= p(z_1 - y + z_2 + y) \\ &\leqslant p(z_1 - y) + p(z_2 + y), \end{aligned}$$

so indeed, $f(z_1) - p(z_1 - y) \leqslant f(z_2) + p(z_2 + y)$. This means that we can simply pick $\bar{f}(x_0)$ to be any element in the closed interval

$$[\sup\{f(z) - p(z - x_0) \mid z \in Z\}, \inf\{p(z + x_0) - f(z) \mid z \in Z\}],$$

which makes sure that $|\bar{f}| \leqslant p$. Linearity and continuity of $\bar{f}$ is automatic. ■

To be able to go from the finite case to the infinite[4], we need some way to "glue" all these uncountably many extensions together. An ultrafilter supplies us with sufficient glue.

Theorem 2.2 (Hahn-Banach; ZF+UL). *Let X be a normed space, $p : X \to \mathbb{R}$ a seminorm, $Z \subseteq X$ a subspace and $f \in Z^*$ a functional satisfying $|f| \leqslant p$. Then there exists a functional $\bar{f} \in X^*$ which extends f and which still satisfies $|\bar{f}| \leqslant p$.*

Proof. Firstly define the set

$$S := \{g \mid \text{"}g \in Y^* \text{ for some subspace } Y \subseteq X \text{ extending } Z\text{"} \wedge |g| \leqslant p)\}.$$

As $f \in S$ it holds that S is nonempty. Define now $A_x := \{g \in S \mid x \in \operatorname{dom} g\}$ for each $x \in X$. Note that the finite extension lemma implies that $A_x \neq \varnothing$ for every $x \in X$ and also that $A_x \cap A_y \neq \varnothing$ for any $x, y \in X$. This means that that A_x's form a subbasis

[4]We can actually still prove Hahn-Banach for countable extensions in ZF, as we can pick e.g. the maximal element of the above interval at each finite stage. This canonical choice allows us to ensure that the sequence exists without resorting to AC.

for a filter on S; i.e. that

$$\mathcal{F} := \{A \subseteq S \mid \exists (x_n)_{n\in\mathbb{N}} : \bigcap_{k\in\mathbb{N}} A_{x_k} \subseteq A\}$$

is a filter. Use UL to get an ultrafilter $\mathcal{U}$ extending $\mathcal{F}$, so that $\{g \in S \mid x \in \operatorname{dom} g\} \in \mathcal{U}$ for all $x \in X$. Form the ultrapower $(\mathcal{M}, \in_{\mathcal{U}}) := (\mathrm{Ult}(V,\mathcal{U}), \in_{\mathcal{U}})$ with associated ultrapower embedding $j : (V, \in) \to (\mathcal{M}, \in_{\mathcal{U}})$ and let $h := [\mathrm{id}]_{\mathcal{U}}$. That $A_x \in \mathcal{U}$ for every $x \in X$ is then equivalent to saying that $j(x) \in_{\mathcal{U}} \operatorname{dom} h$, where

$$\begin{aligned}(\mathcal{M}, \in_{\mathcal{U}}) \models \text{"}h : Y \to j(\mathbb{R}) \text{ for some subspace } Y \subseteq j(X) \\ \text{extending } j(Z)\text{"} \wedge |h| \leqslant j(p).\end{aligned}$$

Now define $\bar{f} \in X^*$ as $\bar{f} := \mathsf{st} \circ h \circ j \restriction X : (X, \in) \to (\mathbb{R}, \in)$. Since $|h(x)| \leqslant j(p)(x)$ for every $x \in j(X)$ we in particular get that

$$\left|\bar{f}(x)\right| = \mathsf{st}(|h(j(x))|) \leqslant \mathsf{st}(j(p)(j(x))) = \mathsf{st}(j(p(x))) = p(x)$$

for $x \in X$. Finally, as

$$\bar{f}(z) = \mathsf{st}(h(j(z))) = \mathsf{st}(j(f)(j(z))) = \mathsf{st}(j(f(z))) = f(z)$$

holds for every $z \in Z$, $\bar{f}$ does indeed extend f, as wanted. ■