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Games and determinacy

Project in Mathematics

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Abstract

In this project we introduce the notions of perfect information games in a set-theoretic context, from where we'll analyse both the consequences of the determinacy of games as well as showing large classes of games are determined.

More precisely, we'll show that determinacy of games over the reals implies that every subset of the reals is Lebesgue measurable and has both the Baire and perfect set property (thereby contradicting the axiom of choice). Next, Martin's result on Borel determinacy will be presented, as well as his proof of analytic determinacy from the existence of a Ramsey cardinal.

Lastly, we'll present a certain kind of stochastic games (that is, games involving chance) called *Blackwell games*, and present Martin's proof that determinacy of perfect information games imply the determinacy of Blackwell games.

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Introduction

Game theory is commonly subdivided into the study of *finite* games and *infinite* games, where economists and computer scientists are primarily interested in the finite case with a focus on analysing the specific strategies of the games, and set theorists being interested in the infinite case where the mere *existence* of winning strategies are of primary interest. We will only work with the infinite case in this project and thus suitably "game theory" will only refer to infinite.

Game theory started with Zermelo's paper [Zer13] on chess in 1913, in which some writers claim that he implicitly proved that chess is *determined*, in the sense that either one of the players has a winning strategy or else both players can force a draw. This was then generalised to the following.

Theorem 0.0.1 (Zermelo). *Every finite two-person game of perfect information is determined, in the sense that if the game doesn't end in a draw, one of the players has a winning strategy.*

Since a winning strategy for the starting player in a finite game can be described as $\forall x_{2n+1} \exists x_{2n} \cdots \forall x_1 \exists x_0 \varphi$ for some formula φ and analogously a winning strategy for the other player can be described as $\exists x_{2n+1} \forall x_{2n} \cdots \exists x_1 \forall x_0 \varphi$, the *determinacy* of the game can then just be seen as de Morgan's law

$$\neg \forall x_n \exists x_{n-1} \cdots \forall x_1 \exists x_0 \varphi \equiv \exists x_n \forall x_{n-1} \cdots \exists x_1 \forall x_0 \varphi.$$

Hence the determinacy of *infinite* games can be seen as an infinite version of de Morgan's law. Using the axiom of choice we can construct a non-determined infinite game, so this infinite de Morgan's law is not true in all generality.

To be able to examine these infinite games more thoroughly we will view games as *trees*, where the starting position in the game is the terminal node and the players alternate in choosing branches. Coding the set of legal moves as natural numbers, we can view the *game tree* as a tree on the natural numbers ω. Putting the discrete topology on ω and the product topology on the *Baire space* $\mathcal{N} := {}^{\omega}\omega$ we can analyse the set of winning moves, which is then a subset of $[T] \subseteq \mathcal{N}$.

Using this kind of machinery, Gale and Stewart proved that both *open* and *closed* games are determined. Martin then generalised this in a substantial way: that every *Borel* game is determined. Harvey Friedman has shown that it's the best possible result in ZFC, in that analytic determinacy requires the existence of an *inaccessible cardinal*, which cannot be proven in ZFC by Gödel's second incompleteness Theorem.

But why are we interested in determinacy of infinite games? This is primarily due to the direct impact it has on the subsets of the real numbers. If we assumed that every game over the reals was determined then every subset of the reals are Lebesgue measurable, has the Baire property as well as the perfect set property. This assumption contradicts the axiom of choice though, so we're interested in studying strong determinacy axioms consistent with choice, such as analytic determinacy.

In this project we'll prove that determinacy entails the above mentioned regularity properties of the reals, prove Martin's subliminal result on the determinacy of Borel games and show that analytic determinacy can be proven if we assume the existence of a so-called *Ramsey cardinal*.

We'll furthermore study games with *imperfect information*, called *Blackwell games*, in which both players play their moves simultaneously. One can intuitively see these games as generalisations of rock-paper-scissors to infinite moves. This requires us to modify our notion of a strategy since we suddenly have to deal with *chance*, and we'll draw on the theory of probability measures in this regard.

As for prerequisites we assume basic knowledge of descriptive set theory, which includes some topology and measure theory. The appendix contains a list of definitions and results used in the project, for reference.

Most notation we're using is standard. However, some things might need clarification: we use round parentheses to denote tuples $(x_0, \ldots, x_n)$ and angled brackets to denote sequences $\langle x_0, \ldots, x_n \rangle$. Furthermore we'll abuse notation and write $s \subseteq t$ for sequences s, t to mean that there exists some $n < \omega$ such that $t \upharpoonright n = s$.

1 BASIC GAME THEORY

1.1 INFINITE GAMES

Let X be any set and $A \subseteq {}^{\omega}X$. To such a pair we can associate a game $G_X(A)$ between two players I and II, where they alternate playing elements $x_i \in X$ for $i < \omega$:

$$\begin{array}{lcccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ \text{II} & & x_1 & & x_3 & & x_5 & & \cdots \end{array}$$

This then results in an element $(x_i) \in {}^{\omega}X$ called a **play** and initial segments of a play are called **partial plays**. For x a play, we write $x_I \in {}^{\omega}X$ for the sequence of player I's moves in x, and likewise $x_{II} \in {}^{\omega}X$ for the sequence of player II's moves in x. We say that player I **wins** if $(x_i) \in A$ and player II wins otherwise - A is called the **payoff set**.

A **strategy** for player I in $G_X(A)$ is a function $\sigma : \bigcup_{n<\omega} {}^{2n}X \to X$ and a strategy for player II is a function $\tau : \bigcup_{n<\omega} {}^{2n+1}X \to X$. We think of a strategy (for e.g. player I) as a way to provide moves for player I given how player II has played so far. Thus If σ is a strategy for player I, the game is played as follows

$$\begin{array}{lcccccc} \text{I} & \sigma(\langle\rangle) & & \sigma(\langle\sigma(\langle\rangle), y_0\rangle) & & \cdots & \\ \text{II} & & y_0 & & y_1 & & \cdots \end{array}$$

We denote such a play by $\sigma * y \in {}^{\omega}X$. A **winning strategy** for player I is a strategy σ for player I such that $\{\sigma * y \mid y \in {}^{\omega}X\} \subseteq A$, i.e. that no matter what player II does, player I will win if he follows the strategy. A winning strategy for player II is defined analogously.

A game is **determined** if one of the players has a winning strategy. We call games of the type $G_X(A)$ **perfect information games**. We say that two games G, G' are **equivalent** if player I has a winning strategy in G iff he has one in G', and player II has a winning strategy in G iff he has one in G'.

Putting the discrete topology on X and the product topology on ${}^\omega X$, we call a game $G_X(A)$ **open, closed, Borel, analytic** etc. if the payoff set $A \subseteq {}^\omega X$ is open, closed, Borel, analytic etc.

We also have a notion of **games with rules**. Letting $T \subseteq {}^\omega X$ be a pruned tree, we restrict the moves allowed by the players to those $x \in X$ such that the partial play is an element of T. For instance, if a partial play looks like $\langle x_0, \dots, x_n \rangle$, then $x \in X$ is a **legal move** if $\langle x_0, \dots, x_n, x \rangle \in T$. Now $G_X(T, A)$ is then the game $G_X(A)$ with legal moves T. If we don't care about the payoff set, then we can leave it out and just write $G_X(T)$.

Strategies σ, τ are defined as before with the extra requirement that $\sigma(s), \tau(t)$ are legal for all s, t. We say $y \in {}^\omega X$ is **σ-legal** if $\sigma * y \in [T]$ and σ is **winning** if $\{\sigma * y \mid \ulcorner y \text{ is } \sigma\text{-legal}\urcorner\} \subseteq A$, and analogously for τ.

It turns out the we're not obtaining a larger class of games by introducing games with rules, because the game $G_X(T, A)$ can be seen to be equivalent to the game $G_X(B)$, where B is defined as the set

$$([T] \cap A) \cup \{x \in {}^\omega X \mid \exists n(x \upharpoonright n \notin T) \wedge \ulcorner \min\{n < \omega \mid x \upharpoonright n \notin T\} \text{ is even}\urcorner\}.$$

1.2 Regularity properties and games

One of the reasons why one might be interested in whether or not games are determined can be seen by how determinacy of certain games entails regularity properties of sets of reals. In this section we'll introduce three games, which are closely related to whether or not a set has the Baire property, the perfect set property or is Lebesgue measurable.

We start off with examining the Baire property.

Definition 1.2.1. Let $A \subseteq \mathcal{N}$. Define the **Banach-Mazur game** $G^{**}_\omega(A)$ as

$$\begin{array}{lcccccccc} \mathrm{I} & s_0 & & s_2 & & s_4 & & \cdots & \\ \mathrm{II} & & s_1 & & s_3 & & s_5 & & \cdots \end{array}$$

where $s_i \in {}^{<\omega}\omega - \{\langle\rangle\}$. Here player I wins iff $s_0 {}^\frown s_1 {}^\frown s_2 {}^\frown \dots \in A$. ○

Lemma 1.2.2 (Banach, Mazur). *Let $A \subseteq \mathcal{N}$. Then in the game $G^{**}_\omega(A)$:*

(i) Player II has a winning strategy iff A is meager.
(ii) Player I has a winning strategy iff there is some $s \in {}^{<\omega}\omega$ such that $N_s - A$ is meager;

Proof. (i) : Assume player II has a winning strategy τ. For each partial play consistent with τ of the form $p = \langle s_0, s_1, \dots, s_{2n-1}\rangle$, set $p_* := s_0 \frown s_1 \frown \cdots \frown s_{2n-1}$ and

$$D_p := \{x \in \mathcal{N} \mid p_* \subseteq x \to (\exists t \in {}^{<\omega}\omega - \{\langle\rangle\})(p_* \frown t \frown \tau(p \frown \langle t\rangle) \subseteq x)\}.$$

Claim 1.2.2.1. D_p is open and dense.

Proof of claim. We have that

$$D_p = \neg N_{p_*} \cup \bigcup_{t \in {}^{<\omega}\omega - \{\langle\rangle\}} N_{p_* \frown t \frown \tau(p \frown \langle t\rangle)},$$

so as the basic open sets are clopen, D_p is open. As for denseness, let $u \in {}^{<\omega}\omega$. If p_* and u are incomparable then $N_u \subseteq D_p$ vacously. If $u \subseteq p_*$ then pick k such that $u \frown \langle k\rangle \nsubseteq p_*$ and thus $N_{u \frown \langle k\rangle} \subseteq D_p$ vacously. If $p_* \subseteq u$ then there is some $t \in {}^{<\omega}\omega - \{\langle\rangle\}$ such that $p_* \frown t = u$. Now pick any $x \in \mathcal{N}$ such that $p_* \frown t \frown \tau(p \frown \langle t\rangle) \subseteq x$ whence $x \in N_u \cap D_p$, making D_p dense. $\diamondsuit$

Hence $\bigcup_p \neg D_p$ is meager. We'll show $A \subseteq \bigcup_p \neg D_p$. If $x \in \bigcap_p D_p$ then we can recursively define a play (s_i) compatible with τ such that $x = s_0 \frown s_1 \frown \cdots$, making $x \notin A$ as τ is winning. Thus $A \subseteq \bigcup_p \neg D_p$, concluding A is meager.

Now assume conversely that A is meager, so $A \subseteq \bigcup_{n<\omega} C_n$ with $C_n \subseteq \mathcal{N}$ being closed and nowhere dense. Then we can define a strategy τ for player II as follows. Set $\tau(\langle s_0\rangle)$ to be some t such that $N_{s_0 \frown t} \cap C_0 = \varnothing$, which is possible as C_0 is nowhere dense. Then recursively set $\tau(\langle s_0, \dots, s_{2n}\rangle)$ to be some t such that $N_{s_0 \frown \dots \frown s_{2n} \frown t} \cap C_n = \varnothing$. It's easily seen that such a strategy is winning for player II.

(ii) : Assume first that player I has a winning strategy σ and set $s := \sigma(\langle\rangle)$. Then it's easily seen that player II has a winning strategy in the game $G^{**}_\omega(N_s - A)$, derived from σ. Hence by (i), $N_s - A$ is meager.

Assume now that there is some $s \in {}^{<\omega}\omega$ such that $N_s - A$ is meager. If $s = \langle\rangle$ then A is comeager and hence A is dense by A.3.4, so player I clearly has a winning strategy. Assume thus $s \neq \langle\rangle$. Then the strategy σ given by $\sigma(\langle\rangle) = s$ and then avoiding $N_s - A$ as we did in (i), is winning. ■

Proposition 1.2.3. *Let $A \subseteq \mathcal{N}$ and define*

$$O_A := \bigcup\{N_s \mid s \in {}^{<\omega}\omega \wedge N_s - A \textit{ is meager}\}.$$

If $G^{**}_\omega(A - O_A)$ is determined then A has the Baire property.

Proof. Assume player I has a winning strategy. Then by Lemma 1.2.2 there is some $s \in {}^{<\omega}\omega$ such that $N_s - (A - O_A)$ is meager. Hence $N_s - A$ is also meager, so that $N_s \subseteq O_A$. But then $N_s - (A - O_A) = N_s$, which cannot be meager, $\lightning$.

Hence player II must have a winning strategy, making $A - O_A$ meager by Lemma 1.2.2. As we also have that $O_A - A$ is meager by definition of O_A and that a union of meager sets is meager, $A =_* O_A$. ■

As for the perfect set property, we turn to a game constructed by Davis.

Definition 1.2.4. Let $A \subseteq \mathcal{C}$. Then define the **Davis game** $G^*_2(A)$ as

$$\begin{array}{lcccccccc} \text{I} & s_0 & & s_1 & & s_2 & & \cdots & \\ \text{II} & & x_0 & & x_1 & & x_2 & & \cdots \end{array}$$

where $s_i \in {}^{<\omega}2$ and $x_i \in 2$. Then player I wins iff $s_0{}^\frown\langle x_0\rangle{}^\frown s_1{}^\frown\langle x_1\rangle{}^\frown \cdots \in A$. ○

Lemma 1.2.5 (Davis). *Let $A \subseteq \mathcal{C}$. Then in the game $G^*_2(A)$:*

(i) Player I has a winning strategy iff A has a perfect subset;;
(ii) Player II has a winning strategy iff A is countable.

PROOF. (i) : Assume σ is a winning strategy for player I. We claim that

$$P := \{\sigma * y \mid y \in \mathcal{C}\}$$

is a perfect subset of A. It's a subset of A since σ is winning. Let $x \in \neg P$. Then there is some $n < \omega$ such that $x \restriction n \neq (\sigma * y) \restriction n$ for every $y \in \mathcal{C}$. But then $N_{x \restriction n} \cap P$ is an open neighborhood around x, so $\neg P$ is open, making P closed.

Lastly let $\sigma * y \in P$ and $N_s \subseteq \mathcal{C}$ an open basis neighborhood around $\sigma * y$. If $\mathrm{len}(s)$ is even then let player II play any other move than in $\sigma * y$ and play arbitrarily following σ, resulting in $\sigma * y' \in P \cap N_s$. If $\mathrm{len}(s)$ is odd then $\mathrm{len}(s \mathbin{^\frown} \langle \sigma(s) \rangle)$ is even, so repeat the previous argument. Hence P is perfect in A.

Assume conversely that $P \subseteq A$ is a perfect subset. Defining

$$T := \{x \restriction n \mid x \in P \wedge n < \omega\},$$

we can form a strategy σ for player I as follows. Set $\sigma(\langle\rangle)$ to be $s \in T$ such that both $s \mathbin{^\frown} \langle 0 \rangle \in T$ and $s \mathbin{^\frown} \langle 1 \rangle \in T$, which can be done since P has no isolated points. In general in response to a partial play p, set $\sigma(p)$ to be $s \in T$ such that both $s \mathbin{^\frown} \langle 0 \rangle \in T$ and $s \mathbin{^\frown} \langle 1 \rangle \in T$. This strategy is easily seen to be winning for player I.

(ii) : Assume τ is a winning strategy for player II. Arguing like in Lemma 1.2.2, for a partial play of the form $p := \langle s_0, x_0, \dots, x_{2n-1} \rangle$ set $p_* := s_0 \mathbin{^\frown} \langle x_0 \rangle \mathbin{^\frown} \cdots \mathbin{^\frown} \langle x_{2n-1} \rangle$ and define

$$D_p := \{x \in \mathcal{C} \mid p_* \subseteq x \to (\exists t \in {}^{<\omega}2)(p_* \mathbin{^\frown} t \mathbin{^\frown} \tau(p \mathbin{^\frown} \langle t \rangle) \subseteq x)\}.$$

As previously, we have that $A \subseteq \bigcup_p \neg D_p$. We'll show that $\neg D_p$ contains exactly one element, making A countable. We have that $x \in \neg D_p$ iff $p_* \subseteq x$ and $(\forall t \in {}^{<\omega}2)(p_* \mathbin{^\frown} t \mathbin{^\frown} \tau(p \mathbin{^\frown} \langle t \rangle) \nsubseteq x)$, so say $\mathrm{len}(p_*) = m$ and we have some $x \in \mathcal{C}$ such that $x \restriction m = p_*$. Now set $i := \tau(p \mathbin{^\frown} \langle \varnothing \rangle) \in 2$. Then we necessarily have that $x(m) = 1 - i$ since otherwise $p_* \mathbin{^\frown} \varnothing \mathbin{^\frown} \tau(p \mathbin{^\frown} \langle \varnothing \rangle) \subseteq x$. Likewise, recursively we must have that

$$x(e) := 1 - \tau(p \mathbin{^\frown} \langle x(m), \dots, x(e-1) \rangle)$$

for $e > m$. Hence $\neg D_p$ has exactly one element x, making A countable.

Now on the other hand assume that A is countable; write $A := \{a_i \mid i < \omega\}$. Player II can now just diagonalize A, playing his i'th move to make sure that that the partial play differs from a_i. This strategy is clearly winning for player II. ■

We have a canonical way of going back and forth between $\mathcal{N}$ and $\mathcal{C}$:

Definition 1.2.6. For $n, k < \omega$ define the functions $b_n^k : n+1 \to 2$ as

$$b_n^{2k}(i) := \begin{cases} 1 & , i < n \\ 0 & , i = n \end{cases} \qquad b_n^{2k+1}(i) := \begin{cases} 0 & , i < n \\ 1 & , i = n \end{cases}$$

Then define the function $\Psi : \mathcal{N} \to \mathcal{C}$ as $\Psi(x) := b^0_{x(0)} {}^\frown b^1_{x(1)} {}^\frown b^2_{x(2)} {}^\frown \cdots$ ○

For instance, if $x = \langle 1, 2, 3, \ldots \rangle$ then

$$\Psi(x) = b_1^0 {}^\frown b_2^1 {}^\frown b_3^2 {}^\frown \ldots = \langle 1, 0, 0, 0, 1, 1, 1, 1, 0, \ldots \rangle,$$

which we also suggestively could write as $1^1 0 0^2 1 1^3 0 \cdots$.

Proposition 1.2.7. *$\Psi : \mathcal{N} \to \mathcal{C}$ is a homeomorphism onto $\mathcal{C}_0$, where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the elements that are not eventually constant.*

Proof. Just as in the example above, notice that every $x \in \mathcal{C}_0$ can be written as

$$x = 1^{\alpha_0} 0 0^{\alpha_1} 1 1^{\alpha_2} 0 \cdots = b^0_{\alpha_0} {}^\frown b^1_{\alpha_1} {}^\frown b^2_{\alpha_2} {}^\frown \cdots$$

for $\alpha_i < \omega$. Define $\Phi : \mathcal{C}_0 \to \mathcal{N}$ given by $\Phi(b^0_{\alpha_0} {}^\frown b^1_{\alpha_1} \cdots) := \langle \alpha_0, \alpha_1, \ldots \rangle$. Then Φ is clearly an inverse to Ψ. It remains to show that both Ψ and Φ are continuous. But we have that

$$\Psi^{-1}(N_{b^0_{\alpha_0} {}^\frown b^1_{\alpha_1} {}^\frown \cdots {}^\frown b^n_{\alpha_n}} \cap \mathcal{C}_0) = N_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle}$$
$$\Phi^{-1}(N_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle}) = N_{b^0_{\alpha_0} {}^\frown b^1_{\alpha_1} {}^\frown \cdots {}^\frown b^n_{\alpha_n}} \cap \mathcal{C}_0,$$

so both are continuous, and Ψ is thus a homeomorphism. ■

Proposition 1.2.8. *For $A \subseteq \mathcal{N}$, $G_2^*(\Psi\text{"}A)$ is determined iff A has the perfect set property.*

Proof. By Lemma 1.2.5, we only have to show that $A \subseteq \mathcal{N}$ is countable iff $\Psi\text{"}A \subseteq \mathcal{C}$ is countable, and $A \subseteq \mathcal{N}$ is perfect iff $\Psi\text{"}A \subseteq \mathcal{C}$ is perfect. But this is clear by the previous Proposition 1.2.7. ■

Lastly we deal with Lebesgue measurability. The game turns out to be more complicated, even though the idea is very simple.

Definition 1.2.9. For $\varepsilon > 0$ and $A \subseteq \mathcal{N}$, define the **covering game** as the game $G(A, \varepsilon)$ given by

$$\begin{array}{lcccccccc} \mathrm{I} & x_0 & & x_1 & & x_2 & & \cdots & \\ \mathrm{II} & & y_0 & & y_1 & & y_2 & & \cdots \end{array}$$

where $x_i \in 2$ and $y_i \in \omega$. Letting $\{s_i \mid i < \omega\} = {}^{<\omega}\omega$ be an enumeration of ${}^{<\omega}\omega$, set $U_i := N_{s_{s_i(0)}} \cup \cdots \cup N_{s_{s_i(\mathrm{len}(s_i)-1)}}$. We require of player I that (x_i) is not eventually constant and of player II that $\lambda(U_{y_i}) < \varepsilon / 2^{2(i+1)}$ – if either player cannot satisfy this, he loses. If this doesn't happen then player I wins iff $\Psi^{-1}(x) \in A - \bigcup_{n<\omega} U_{y_n}$. I.e. player II tries to cover A with $\bigcup_{n<\omega} U_{y_n}$. ○

Lemma 1.2.10. *Let $A \subseteq \mathcal{N}$ and $\varepsilon > 0$. Then in the covering game $G(A, \varepsilon)$:*

(i) If player I has a winning strategy then there's a Lebesgue measurable $B \subseteq A$ with positive Lebesgue measure;

(ii) If player II has a winning strategy then there's an open set $O \supseteq A$ such that $\lambda(O) < \varepsilon$.

Proof. (i) : Let σ be a winning strategy for player I. Define

$$B := \{\Psi^{-1}((\sigma * y)_I) \mid y \in \mathcal{N}\}.$$

Since σ is winning, $\Psi^{-1}((\sigma * y)_I) \in A$ for every $y \in \mathcal{N}$, so $B \subseteq A$. Define the function $f : \mathcal{N} \to \mathcal{N}$ as $f(y) := \Psi^{-1}((\sigma * y)_I)$. We claim that f is continuous. It's enough to check that $f_n : \mathcal{N} \to 2$ given by $f_n(y) := (\sigma * y)_I(n) = \sigma((\sigma * y) \upharpoonright 2n)$

is continuous for every $n < \omega$. And indeed, we have that

$$\begin{aligned} f_n^{-1}(\{k\}) &= \{y \in \mathcal{N} \mid (\sigma * y)_I(n) = k\} \\ &= \bigcup \{N_s \mid s \in {}^{<\omega}\omega \wedge \operatorname{len}(s) = 2n \wedge \ulcorner s \text{ follows } \sigma \urcorner \wedge \sigma(s) = k\} \end{aligned}$$

is open, so f_n is continuous, making f continuous as well. This means that B is a continuous image of the Baire space, making it analytic and hence Lebesgue measurable by Theorem A.5.4.

It remains to show that B has positive Lebesgue measure. Assume B is λ-null. By Lemma A.5.2, there exists an open $O \supseteq B$ such that $\lambda(O) < \varepsilon/4 = \varepsilon/2^{2(0+1)}$. Since $O = \coprod_{i<\omega} N_{s_i}$ by Proposition A.1.7, cut the N_{s_i}'s into smaller basic opens to wind up with a countable collection of basic opens $(N_{s_j})_{j<\omega}$ such that $O = \coprod_{j<\omega} N_{s_j}$ and $\lambda(N_{s_j}) < \varepsilon/2^{2(j+1)}$ for every $j < \omega$. Then the $y \in \mathcal{N}$ coding $(N_{s_j})_{j<\omega}$ is a winning strategy for player II, $\lightning$. Hence $\lambda(B) > 0$.

(ii) : Let τ be a winning strategy for player II. For any $s \in {}^{<\omega}2 - \{\langle\rangle\}$, say with $\operatorname{len}(s) = n+1$, set $d(s) \in X$ to be player II's response to the partial play consistent with τ where player I's moves are $s(0), \dots, s(n)$. Since τ is winning then given any $x \in A$ there exists some $n < \omega$ such that $x \in U_{d(\Psi(x) \restriction n+1)}$. Thus, defining

$$O := \bigcup \{U_{d(s)} \mid s \in {}^{<\omega}\omega - \{\langle\rangle\}\},$$

we see that $A \subseteq O$. The rules of game require that $\lambda(U_{d(s)}) < \varepsilon/2^{2(n+1)}$ for $\operatorname{len}(s) = n+1$, so since there are 2^{n+1} such s's, we have that

$$\lambda(O) < \sum_{n<\omega} \frac{\varepsilon}{2^{2(n+1)}} \cdot 2^{n+1} = \sum_{n<\omega} \frac{\varepsilon}{2^{n+1}} = \varepsilon.$$

■

Definition 1.2.11. Let $A, B \subseteq \mathcal{N}$. Then B is a **minimal cover** for A if $A \subseteq B$, B is Lebesgue measurable and if $Z \subseteq B - A$ is a Lebesgue measurable set then $\lambda(Z) = 0$. ○

Notice that any $A \subseteq \mathcal{N}$ has a minimal G_δ cover by Lemma A.5.2.

Proposition 1.2.12. *Let $A \subseteq \mathcal{N}$ and let $B \subseteq \mathcal{N}$ be a minimal G_δ cover for A. If $G(B - A, \varepsilon)$ is determined for every $\varepsilon > 0$ then A is Lebesgue measurable.*

Proof. Assume $G(B - A, \varepsilon)$ is determined for every $\varepsilon > 0$. By Lemma 1.2.10, it's impossible for player I to have a winning strategy as B is a minimal cover for A. Hence by the same Lemma there exist open sets $O_\varepsilon \supseteq B - A$ with $\lambda(O_\varepsilon) < \varepsilon$ for every $\varepsilon > 0$. Defining

$$C := \bigcap_{n<\omega} O_{1/n},$$

we see that $B - A \subseteq C$ and $\lambda(C) = 0$. Hence $A \triangle B$ is a λ-null set, so since B is G_δ and in particular Borel, A is Lebesgue measurable. ■

1.3 Axiom of determinacy

Mycielski and Steinhaus proposed the following axiom in [MS62].

Definition 1.3.1. The **Axiom of Determinacy**, AD, is the statement that $G_\omega(A)$ is determined for every $A \subseteq \mathcal{N}$, i.e. that every perfect information game of reals is determined. ○

Theorem 1.3.2 (ZF). *AD implies that every subset of the reals is Lebesgue measurable and has both the Baire property and the perfect set property.*

Proof. This follows by Propositions 1.2.3, 1.2.8 and 1.2.12 if we can show that the three corresponding games are equivalent to a game of the form $G_\omega(B)$. We start off with the Banach-Mazur game $G^{**}_\omega(A)$, where each player playes sequences $s_i \in {}^{<\omega}\omega - \{\langle\rangle\}$ and player I won iff $s_0{}^\frown s_1{}^\frown \cdots \in A$. By fixing an enumeration $\{s_i \mid i < \omega\}$ of ${}^{<\omega}\omega - \{\langle\rangle\}$, the players can just play $x_i < \omega$ instead and setting $B := \{x \in \mathcal{N} \mid s_{x(0)}{}^\frown s_{x(1)}{}^\frown \cdots \in A\}$, $G^{**}_\omega(A)$ is equivalent to $G_\omega(B)$.

Now for the Davis game $G^*_2(A)$, where player I played $s_i \in {}^{<\omega}2$, player II played $x_i \in 2$ and player I won iff $s_0{}^\frown\langle x_0\rangle{}^\frown s_1{}^\frown\langle x_1\rangle{}^\frown \cdots \in A$. By fixing an enumeration $\{s_i \mid i < \omega\}$ of ${}^{<\omega}2$, player I and II can play $x_i < \omega$ and $y_i < \omega$, respectively, and setting

$$B := \{x \in \mathcal{N} \mid s_{x_0} {}^\frown \langle \Psi(x_1) \rangle {}^\frown s_{x_2} {}^\frown \langle \Psi(x_3) \rangle {}^\frown \cdots \in A\},$$

we see that $G_2^*(A)$ is equivalent to $G_\omega(B)$. Hence we also have that $G_2^*(\Psi\text{"}A)$ is equivalent to $G_\omega(\Psi\text{"}B)$.

Lastly, for the covering game $G(A, \varepsilon)$ we can just make player I play $x_i \in \omega$ instead and say that player II wins iff $x \in A - \bigcup_{n<\omega} N_{y_n}$. ■

We see that AD implies a lot of nice properties about the real numbers, but these properties turns out to be a bit *too* nice: AD contradicts the axiom of choice.

Corollary 1.3.3. *AC implies that AD is false.*

Proof. As AC allows us to construct a set not having the Baire property by Proposition A.3.5, we have that ZF $\vdash$ AC $\to \neg$ AD. ■

Hence we can see AD as removing the “non-regular” sets of reals. AD and AC are not completely inconsistent however; we're allowed to accept *countable* choice over the reals $\mathsf{AC}_\omega(\mathcal{N})$, stating that every countable set of non-empty sets of reals has a choice function.

Proposition 1.3.4 (ZF). *AD implies $\mathsf{AC}_\omega(\mathcal{N})$.*

Proof. Let $\{X_n \mid n < \omega\} \subseteq \mathcal{P}(\mathcal{N}) - \{\varnothing\}$. Define

$$A := \{x \in \mathcal{N} \mid x_{II} \notin X_{x(0)}\}.$$

Clearly player I cannot have a winning strategy in $G_\omega(A)$, so AD implies that we have a winning strategy τ for player II. Define now $f : \omega \to \mathcal{N}$ given by $f(n) := (\langle n, 0, 0, \ldots \rangle * \tau)_{II}$. It's easy to check that f is a choice function. ■

2 Borel determinacy

Now that we've seen several positive consequences of the determinacy of games, the natural question is then whether or not we can actually find a large class of determined games.

2.1 Determinacy of open and closed games

Definition 2.1.1. A **quasistrategy** Σ in $G_X(T)$ is a nonempty set of strategies. For $y \in {}^\omega X$ we define $\Sigma * y$ as the set of legal plays of the form

$$\begin{array}{lcccccc} \mathrm{I} & \sigma_0(\langle\rangle) & & \sigma_1(\langle\sigma_0(\langle\rangle), y_0\rangle) & & \cdots & \\ \mathrm{II} & & y_0 & & y_1 & & \cdots \end{array}$$

where $\sigma_i \in \Sigma$ for every $i < \omega$. A quasistrategy Σ in $G_X(T, A)$ is said to be **winning** for player I if $\bigcup_{y \in {}^\omega X} \Sigma * y \subseteq A$. Winning quasistrategies for player II are defined analogously. ○

Say that a position $\langle x_0, \dots, x_{2n-1}\rangle \in T$ in $G_X(T, A)$ is **not losing** for player I if player II has no winning strategy from that point on. Define the **canonical quasistrategy** for player I as

$$\Sigma := \{\sigma \mid \ulcorner \sigma \text{ is a strategy for player I} \urcorner \wedge \forall s \in T : \ulcorner \sigma(s) \text{ is not losing} \urcorner\},$$

and analogously for player II.

Lemma 2.1.2.

(i) If $G_X(T, A)$ is a closed game in which player II has no winning strategy, then the canonical quasistrategy for player I in $G_X(T, A)$ is a winning quasistrategy;

(ii) If $G_X(T, A)$ is an open game in which player I has no winning strategy, then the canonical quasistrategy for player II in $G_X(T, A)$ is a winning quasistrategy.

Proof. (i): Let Σ be the canonical quasistrategy for player I in $G_X(T, A)$. We will show that (a) $\Sigma \neq \varnothing$ and (b) that every strategy $\sigma \in \Sigma$ is winning.

To show (a), note first that $\langle\rangle \in T$ is not losing for player I by assumption. Assume now that player I is not losing at $p := \langle x_0, \dots, x_{2n-1}\rangle \in T$. Then there exists some $x_{2n} \in X$ such that for every $x_{2n+1} \in X$, $p^\frown\langle x_{2n}, x_{2n+1}\rangle$ is not losing for player I.

Indeed, assuming it wasn't the case we would have that no matter what player I played, player II would have a play such that player II would have a winning strategy at that point. But this means exactly that player II had a winning strategy at p, so p would be losing for player I, $\lightning$. Hence the strategy just described is an element of Σ, so (a) is shown.

Now, to show (b), let $\sigma \in \Sigma$ and $y \in {}^\omega X$. By definition of Σ, $(\sigma * y) \upharpoonright 2k$ is not losing for player I, for every $k < \omega$. Assume for a contradiction that $\sigma * y \notin A$. Then since A is closed, $\neg A$ is open, so there is some open neighborhood $N_q \cap [T]$ in $[T]$ around $\sigma * y$ such that $\mathrm{len}(q)$ is even and $N_q \cap [T] \cap A = \varnothing$. But since $q = (\sigma * y) \upharpoonright 2k$ for some $k < \omega$, we have that q is both losing and not losing for player I, $\lightning$. Hence $\sigma * y \in A$ and σ is winning. Hence (b) is shown.

The argument for (ii) is completely analogous. ■

Theorem 2.1.3 (Gale-Stewart). *Every open or closed game is determined.*

Proof. If player II has no winning strategy in the closed game $G_X(T, A)$, the canonical quasistrategy Σ for player I in $G_X(T, A)$ is winning by Lemma 2.1.2, so any $\sigma \in \Sigma$ is a winning strategy for player I in $G_X(T, A)$. The argument is analogous for open games, by swapping player I and player II, using Lemma 2.1.2 again. ■

2.2 Borel determinacy

Now we'll work towards Martin's subliminal result stating that every Borel game is determined. To make things easier we'll introduce the notion of a *covering*.

Definition 2.2.1. A **covering** $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ is a pair (π_f, φ_f) such that

(i) $\pi_f : \tilde{T} \to T$ is monotone and length-preserving, so that we have a continuous extension $\tilde{\pi}_f : [\tilde{T}] \to [T]$;

(ii) φ_f maps partial strategies $\sigma \restriction n$ to partial strategies $\varphi_f(\sigma \restriction n)$ satisfying that if $m \leqslant n$ then $\varphi_f(\sigma \restriction m) = \varphi_f(\sigma \restriction n) \restriction m$, which gives rise to an extension $\tilde{\varphi}_f$ defined on all strategies in $G_{\tilde{X}}(\tilde{T})$, given by $\tilde{\varphi}_f(\sigma) \restriction n = \varphi_f(\sigma \restriction n)$;

(iii) If $x \in T$ is played according to a strategy $\tilde{\varphi}_f(\sigma)$ then there's an $\tilde{x} \in \tilde{T}$ played according to σ so that $\tilde{\pi}_f(\tilde{x}) = x$.

$\circ$

Note that we can compose coverings $f : G_X(T_X) \to G_Y(T_Y)$ and $g : G_Y(T_Y) \to G_Z(T_Z)$ simply by defining $g \circ f := (\pi_g \circ \pi_f, \varphi_g \circ \varphi_f)$. It's clear that the composition is still a covering.

Definition 2.2.2. A covering $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ **unravels** $A \subseteq [T]$ if $\tilde{\pi}_f^{-1}(A)$ is clopen in $[\tilde{T}]$. A game $G_X(T, A)$ is **unraveled** if there exists a covering $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ unraveling A. $\circ$

Unraveling coverings grants us with a sufficient condition for a game to be determined, which is also considerably easier to work with.

Proposition 2.2.3. *Every unraveled game $G_X(T, A)$ is determined.*

Proof. Let $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ be a covering unraveling A. First of all $G_{\tilde{X}}(\tilde{T}, \tilde{\pi}_f^{-1}(A))$ is determined by the Gale-Stewart Theorem 2.1.3, so we have to show that if σ is a winning strategy in $G_{\tilde{X}}(\tilde{T}, \tilde{\pi}_f^{-1}(A))$ then $\tilde{\varphi}_f(\sigma)$ is a winning strategy in $G_X(T, A)$.

Let x be a play in $G_X(T, A)$ according to $\tilde{\varphi}_f(\sigma)$. Since f is a covering, there is some $\tilde{x} \in \tilde{T}$ played according to σ such that $\tilde{\pi}_f(\tilde{x}) = x$. Then $\tilde{x} \in \tilde{\pi}_f^{-1}(A)$ as σ is winning, so $x = \tilde{\pi}_f(\tilde{x}) \in A$. Hence $\tilde{\varphi}_f(\sigma)$ is winning and $G_X(A)$ is determined. ■

We'll define a stronger notion of a k-**covering**, which we'll need in the proof of Borel determinacy to carry out an inductive argument.

Definition 2.2.4. A covering $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ is a k-**covering** if $\tilde{T} \upharpoonright 2k = T \upharpoonright 2k$ and $\pi_f \upharpoonright (\tilde{T} \upharpoonright 2k) = \mathrm{id} \upharpoonright (T \upharpoonright 2k)$. A game $G_X(T, A)$ is k-**unraveled** if there exists a k-covering $f : G_{\tilde{X}}(\tilde{T}) \to G_X(T)$ unraveling A. ◦

Now we'll begin with the actual argument. The substantial bit of the argument will be the following lemma, stating that we can k-unravel closed games.

Lemma 2.2.5. *Every closed game $G_X(T, A)$ is k-unraveled for every $k < \omega$.*

Proof. Fix a closed $G_X(T, A)$ and $k < \omega$. For a tree S, set $S_u := \{v \mid u \frown v \in S\}$ and for $Y \subseteq [S]$, set $Y_u := \{x \mid u \frown x \in Y\}$. Define

$$T_A := \{s \in T \mid \exists a \in A : s \subseteq a\}.$$

We're going to define a k-covering $G_{\tilde{X}}(\tilde{T})$ of $G_X(T)$ by modifying a few of the rules of the game. Since it's going to be a k-covering, we're forced to say that $\tilde{T} \upharpoonright 2k = T \upharpoonright 2k$. However, at the $2k$'th turn player I plays a pair (x_{2k}, Σ_I):

$$\begin{array}{lccccccc} \mathrm{I} & x_0 & & \cdots & & x_{2k-2} & & (x_{2k}, \Sigma_I) \\ \mathrm{II} & & x_1 & & \cdots & & x_{2k-1} & \end{array}$$

where $x_{2k} \in X$ and Σ_I is a quasistrategy for player I in the game $G_X(T_{\langle x_0,\ldots,x_{2k}\rangle})$, where we without loss of generality assume that in the game $G_X(T_{\langle x_0,\ldots,x_{2k}\rangle})$ player II starts. Now player II can play moves of two different kinds:

$$\begin{array}{lcccccccc} \mathrm{I} & x_0 & & \cdots & & (x_{2k}, \Sigma_I) & & x_{2k+2} & \\ \mathrm{II} & & x_1 & & \cdots & & (x_{2k+1}, u) & & x_{2k+3} \end{array}$$

The **first choice** is to play (x_{2k+1}, u), where $x_{2k+1} \in X$ and $u \in T_{\langle x_0,\ldots,x_{2k+1}\rangle}$ is a sequence of even length following Σ_I such that $u \notin (T_A)_{\langle x_0,\ldots,x_{2k+1}\rangle}$. Furthermore we require that future moves by either player are consistent with u, i.e. that $u \subseteq \langle x_{2k+2}, x_{2k+3}, \ldots\rangle$. In other words, player II follows player I's quasistrategy, but ensures that the play will land outside A.

$$\begin{array}{lcccccccc} \mathrm{I} & x_0 & & \cdots & & (x_{2k}, \Sigma_I) & & x_{2k+2} & \\ \mathrm{II} & & x_1 & & \cdots & & (x_{2k+1}, \Sigma_{II}) & & x_{2k+3} \end{array}$$

The **second choice** is to play (x_{2k+1}, Σ_{II}), where $x_{2k+1} \in X$ and $\Sigma_{II} \subseteq \Sigma_I$ is a quasistrategy for player II in the game $G_X(T)$ with $\{x * \Sigma_{II} \mid x \in {}^\omega X\} \subseteq A$ and $\tau(\langle\rangle) = x_{2k+1}$ for every $\tau \in \Sigma_{II}$. Furthermore we require that future moves by either player are consistent with Σ_{II}. In other words, player II follows player I's quasistrategy and ensures that the play will land *in* A.

Now say that $\tilde{T}$ is the tree of all the legal moves in the game just described, and since it's clear that every player has a valid move at every step of the game, $\tilde{T}$ is pruned. We define $\pi_f : \tilde{T} \to T$ as

$$\pi_f(\langle x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, \square), x_{2k+2}, \ldots, x_l\rangle) := \langle x_0, \ldots, x_l\rangle,$$

where $\square$ is either of the form u or Σ_{II} as described above. Notice that

$$\tilde{x} \in \tilde{\pi}_f^{-1}(A) \Leftrightarrow \tilde{x}(2k+1) \text{ is of the form } (x_{2k+1}, \Sigma_{II}),$$

so $\tilde{\pi}_f^{-1}(A)$ and $\neg\tilde{\pi}_f^{-1}(A)$ are clearly open, making $\tilde{\pi}_f^{-1}(A)$ clopen. Hence it only remains to define φ_f and make sure that it's compatible with π_f. This turns out to involve a lot of book-keeping, so buckle up.

Let $\tilde{\sigma}$ be a strategy in $G_{\tilde{X}}(\tilde{T})$. We'll describe the strategy $\sigma := \tilde{\varphi}_f(\tilde{\sigma})$ informally, and it'll follow by construction that for every play x in $G_X(T)$ played according to $\tilde{\varphi}_f(\tilde{\sigma})$ there's a play $\tilde{x}$ in $G_{\tilde{X}}(\tilde{T})$ such that $\tilde{\pi}_f(\tilde{x}) = x$. We split into the cases where $\tilde{\sigma}$ is a strategy for player I and player II, respectively.

Case 1: $\tilde{\sigma}$ is a strategy for player I.

First of all σ follows $\tilde{\sigma}$ for the first $2k$ moves. Then $\tilde{\sigma}$ provides player I with (x_{2k}, Σ_I); set σ's move to be x_{2k}. Now player II plays some x_{2k+1} in $G_X(T)$. To determine what σ should respond to this move, we'll split into two subcases.

The **first subcase** is if player I has a winning strategy in the game

$$G_X(T^{(I)}_{\langle x_0,\ldots,x_{2k+1}\rangle}, \neg A_{\langle x_0,\ldots,x_{2k+1}\rangle}), \tag{†}$$

where $T^{(I)}_{\langle x_0,\ldots,x_{2k+1}\rangle} \subseteq T_{\langle x_0,\ldots,x_{2k+1}\rangle}$ is the subtree consisting of all the moves consistent with Σ_I. In this subcase, we set σ to follow this winning strategy. Then after finitely many moves we arrive at a minimal position u of even length satisfying

$u \notin (T_A)_{\langle x_0,\dots,x_{2k+1}\rangle}$, say $u = \langle x_{2k+2}, \dots, x_{2l-1}\rangle$. Then

$$\langle x_0, \dots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \dots, x_{2l-1}\rangle$$

is a legal position in $G_{\tilde{X}}(\tilde{T})$. From here σ stops following the winning strategy and instead follows $\tilde{\sigma}$ again.

The **second subcase** is if player II has a winning strategy in the game (†). Let Σ_{II} be the canonical quasistrategy for player II in (†) (this is where we use that $A_{\langle x_0,\dots,x_{2k+1}\rangle}$ is closed by assumption). Then just set σ to follow $\tilde{\sigma}$'s moves under the assumption that player II played (x_{2k}, Σ_{II}) in $G_{\tilde{X}}(\tilde{T})$.

This requires player II's cooperation however, since we have to make sure that player II's moves are legal moves in $G_{\tilde{X}}(\tilde{T})$, i.e. that they are consistent with Σ_{II}. If player II deviates from Σ_{II} then by definition of Σ_{II} player I has a winning strategy in (†), so we can define σ as in the first subcase.

Case 2: $\tilde{\sigma}$ is a strategy for player II.

Just as before, σ follows $\tilde{\sigma}$ for the first $2k$ moves. Then player I plays some x_{2k} in $G_X(T)$. Again we want to split into the subcases corresponding to player II's choice between moves of the form (x_{2k+1}, u) and (x_{2k+1}, Σ_{II}) in $G_{\tilde{X}}(\tilde{T})$, but we have to choose what Σ_I is this time around.

Define $U \subseteq T_{\langle x_0,\dots,x_{2k}\rangle}$ as $s \in U$ iff $s = \langle x_{2k+1}\rangle ^\frown u$ such that $x_{2k+1} \in X$, u has even length and there is a quasistrategy Σ_I for player I in $G_X(T_{\langle x_0,\dots,x_{2k}\rangle})$ such that $\tilde{\sigma}$ requires player II to play (x_{2k+1}, u) when player I plays (x_{2k}, Σ_I). Define

$$\mathcal{U} := \{x \in [T_{\langle x_0,\dots,x_{2k}\rangle}] \mid \exists s \in U : s \subseteq x\},$$

which is easily seen to be an open set in $[T_{\langle x_0,\dots,x_{2k}\rangle}]$. Now we'll use $\mathcal{U}$ to decide for us what player I should play. Define the game $G_X(T_{\langle x_0,\dots,x_{2k}\rangle}, \neg\mathcal{U})$ in which player II plays first and player II wins iff $\langle x_{2k+1}, x_{2k+2}, \dots\rangle \in \mathcal{U}$.

The **first subcase** is if player II has a winning strategy in this game. Then σ follows this winning strategy for player II in $G_X(T)$ until $\langle x_{2k+1}, \dots, x_{2l-1}\rangle \in U$. Then set $u := \langle x_{2k+2}, \dots, x_{2l-1}\rangle$ and let Σ_I witness the fact that the sequence is in U. Now from here on σ can just follow $\tilde{\sigma}$ from

$$\langle x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \ldots, x_{2l-1}\rangle.$$

The **second subcase** is if player I has a winning strategy in this game. Since $\mathcal{U}$ was open, this game is a closed game for player I, so let Σ_I be the canonical strategy for player I. Then if player I plays (x_{2k}, Σ_I) in $G_{\tilde{X}}(\tilde{T})$, player II cannot play anything of the form (x_{2k+1}, u) since then $(x_{2k+1})^\frown u \in U$ is consistent with Σ_I, contradicting the definition of Σ_I.

This means that if player I plays (x_{2k}, Σ_I) in $G_{\tilde{X}}(\tilde{T})$, player II *must* respond with a move of the form (x_{2k+1}, Σ_{II}). Let σ play this x_{2k+1} and then just follow $\tilde{\sigma}$ from

$$\langle x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, \Sigma_{II})\rangle.$$

Just as before though, this requires player I's cooperation. If player II plays some x_{2l} not consistent with Σ_{II} then since Σ_{II} is a quasistrategy for player II and $\Sigma_{II} \subseteq \Sigma_I$, x_{2l} is inconsistent with Σ_I as well, so we're back in the first subcase again. ■

Before we move on to the proof of the theorem we need the following technical lemma, which is the reason why we need to work with *k-coverings* instead of regular coverings.

Lemma 2.2.6 (Existence of inverse limits). *Let $k < \omega$ and $f_{i+1} : G_{X_{i+1}}(T_{i+1}) \to G_{X_i}(T_i)$ a $(k+i)$-covering for every $i < \omega$. Then there's a game $G_X(T)$ and $(k+i)$-coverings $F_i : G_X(T) \to G_{X_i}(T_i)$ for every $i < \omega$ such that $f_{i+1} \circ F_{i+1} = F_i$.*

Proof. We need to define X, T and π_{F_i}, φ_{F_i} for every $i < \omega$. Set $X := \bigcup_{i<\omega} X_i$ and

$$T := \bigcup_{i<\omega} T_i \upharpoonright 2(k+i).$$

This is a pruned tree since $T_j \upharpoonright 2(k+i) = T_i \upharpoonright 2(k+i)$ for every $j \geqslant i$, so the trees in the union "agree" in their intersections. Furthermore it's clear that $T \upharpoonright 2(k+i) = T_i \upharpoonright 2(k+i)$ holds for every $i < \omega$.

Now for π_{F_i}, set $\pi_{F_i}(s) := s$ if $\mathrm{len}(s) \leqslant 2(k+i)$ and if $\mathrm{len}(s) > 2(k+i)$ pick some $j < \omega$ such that $\mathrm{len}(s) \leqslant 2(k+j)$ and set

$$\pi_{F_i}(s) := \pi_{f_{i+1}} \circ \pi_{f_{i+2}} \circ \cdots \circ \pi_{f_j}(s).$$

The choice of j doesn't matter, since $\pi_{f_j}(s) = s$ for every j such that $\mathrm{len}(s) \leqslant 2(k+j)$. Lastly we define φ_{F_i}. Let σ be a strategy in $G_X(T)$ and define $\varphi_{F_i}(\sigma) \upharpoonright 2(k+i) := \sigma \upharpoonright 2(k+i)$ and for $j > i$ set

$$\varphi_{F_i}(\sigma) \upharpoonright 2(k+j) := \varphi_{f_{i+1}} \circ \cdots \circ \varphi_{f_j}(\sigma \upharpoonright 2(k+j)).$$

Note that this a well-defined partial strategy as $T_j \upharpoonright 2(k+j) = T \upharpoonright 2(k+j)$. As the F_i's clearly satisfy conditions (i) and (ii) in the definition of a covering and $f_{i+1} \circ F_{i+1} = F_i$ holds by construction, we only need to check condition (iii), i.e. that π_{F_i} and φ_{F_i} are compatible with each other.

Fix some $i < \omega$, let σ be a strategy in $G_X(T)$ and let $x_i \in [T_i]$ be a play according to $\tilde{\varphi}_{F_i}(\sigma)$. Since $f_{i+j+1} : G_{X_{i+j+1}}(T_{i+j+1}) \to G_{X_{i+j}}(T_{i+j})$ is a $(k+i+j)$-covering by assumption for all $j < \omega$, pick $x_{i+j+1} \in [T_{i+j+1}]$ compatible with $\varphi_{F_{i+j+1}}(\sigma)$ such that $\tilde{\pi}_{i+j}(x_{i+j+1}) = x_{i+j}$ – this is possible since $\varphi_{F_{i+j}} = \varphi_{f_{i+j+1}} \circ \varphi_{F_{i+j+1}}$.

Now the sequence $(x_{i+j+1})_{j<\omega}$ converges to some $x \in T$ given by $x \upharpoonright 2(k+i+j) = x_{i+j} \upharpoonright 2(k+i+j)$ since $\pi_{f_{i+j+1}}$ is the identity on sequences of length $\leqslant 2(k+i+j)$. Since $\varphi_{F_{i+j}}(\sigma) \upharpoonright 2(k+i+j) = \sigma \upharpoonright 2(k+i+j)$ and x_{i+j+1} is compatible with $\varphi_{F_{i+j+1}}(\sigma)$, $x \upharpoonright 2(k+i+j)$ is compatible with σ for every $j < \omega$, meaning x is compatible with σ as well. Finally, we clearly have $\pi_{F_i}(x) = x_i$. ■

The game $G_X(T)$ in the above lemma is called the **inverse limit** of the games $G_{X_n}(T_n)$ and is denoted by $\varprojlim_n G_{X_n}(T_n)$. We now have completed our preparation and we'll move on to proving Martin's theorem.

Theorem 2.2.7 (Martin). *Any Borel game $G_X(A)$ is determined.*

Proof. By definition of the Borel hierachy and by Proposition 2.2.3, it suffices to show that all $\mathbf{\Sigma}^0_\xi$ games $G_X(T, A)$ are unraveled for any $1 \leqslant \xi < \omega_1$. We will show something slightly stronger, which is that they are not only unraveled, but k-unraveled for any $k < \omega$.

Since closed games are k-unraveled for any $k < \omega$ by Lemma 2.2.5 and as $G_X(T, \neg A)$ is unraveled iff $G_X(T, A)$ is, $\mathbf{\Sigma}^0_1$ games are k-unraveled.

Assume now that $\mathbf{\Sigma}^0_\eta$ games and thus also $\mathbf{\Pi}^0_\eta$ games are k-unraveled for every $\eta < \xi$. Let $k < \omega$ and a $\mathbf{\Sigma}^0_\xi$ game $G_X(T, A)$ be given. By definition, $A = \bigcup_{n<\omega} A_n$ for $A_n \in \bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta$. Let $f_0 : G_{X_1}(T_1) \to G_{X_0}(T)$ be a k-covering unraveling A_0 where $X_0 := X$, and recursively let

$$f_i : G_{X_{i+1}}(T_{i+1}) \to G_{X_i}(T_i)$$

be a $(k+i)$-covering unraveling $\tilde{\pi}^{-1}_{f_{i-1}} \circ \cdots \circ \tilde{\pi}^{-1}_{f_1}(A_i)$, which exists as $\mathbf{\Pi}^0_\eta$ is closed under continuous preimages for $\eta < \xi$.

Now let $G_{\tilde{X}}(\tilde{T}) := \varprojlim_n G_{X_n}(T_n)$ be the inverse limit as given by Lemma 2.2.6, with corresponding coverings F_i. Then $F_0 : G_{\tilde{X}}(\tilde{T}) \to G_X(T_0)$ is a k-covering unraveling A_0 as $\tilde{\pi}^{-1}_{F_0}(A_0) = \tilde{\pi}^{-1}_{F_1} \circ \tilde{\pi}^{-1}_{f_1}(A_0)$, $\tilde{\pi}^{-1}_{f_1}(A_0)$ is clopen by assumption and $\tilde{\pi}_{F_0}$ is continuous. By a similar argument F_0 unravels every A_i.

Hence $\tilde{\pi}^{-1}_{F_0}(A) = \bigcup_n \tilde{\pi}^{-1}_{F_0}(A_n)$ is open, so let $\tilde{F} : G_Y(T_Y) \to G_{\tilde{X}}(\tilde{T})$ be a k-covering unraveling $\tilde{\pi}^{-1}_{F_0}(A)$ given by Lemma 2.2.5. But then

$$F_0 \circ \tilde{F} : G_Y(T_Y) \to G_X(T)$$

is a k-covering unraveling A and we're done. ■

3 Analytic determinacy

We now move on to the natural next step: to prove determinacy of analytic games. However, as we'll show, this turns out to be unprovable in ZFC, so we'll prove it in the presence of a large cardinal, called a *Ramsey cardinal*.

3.1 Ramsey cardinals

Ramsey cardinals are cardinals defined in a combinatorial context, generalising a classical theorem of Ramsey. It will be defined in terms of a *partition property*, which is defined as follows.

Definition 3.1.1. The statement $\beta \to (\alpha)^\gamma_\delta$ means that every function (also called a *colouring*) $c : [\beta]^\gamma \to \delta$ has a *homogeneous* set $H \in [\beta]^\alpha$, meaning that $|c"[H]^\gamma| \leqslant 1$. Furthermore $\beta \to (\alpha)^{<\omega}_\delta$ means that given $c : [\beta]^{<\omega} \to \delta$, there's a set $H \in [\beta]^\alpha$ which is homogeneous for every $c \restriction [\beta]^n$. ○

Note that if H is a homogeneous set for $c : [\kappa]^{<\omega} \to \delta$, we might have that $c"[\kappa]^n \neq c"[\kappa]^m$ for $m \neq n$.

Definition 3.1.2. A cardinal κ is **Ramsey** if $\kappa \to (\kappa)^{<\omega}_2$. ○

The reason why the cardinals are called Ramsey is due to *Ramsey's Theorem*, stating that $\omega \to (\omega)^n_k$ holds for all $n, k < \omega$. Note that this doesn't imply that ω is Ramsey, however.

Proposition 3.1.3. *κ is Ramsey iff $\kappa \to (\kappa)^{<\omega}_\delta$ for every $\delta < \kappa$.*

Proof. " $\Leftarrow$ " is clear. Assume thus that $\kappa \to (\kappa)^{<\omega}_2$ and $c : [\kappa]^{<\omega} \to \delta$ a colouring with $\delta < \kappa$. Define $\tilde{c} : [\kappa]^{<\omega} \to 2$ by

$$\tilde{c}(\xi_1, \dots, \xi_n) := \begin{cases} 0 & , (\exists m < \omega)(n = 2m \wedge c(\xi_1, \dots, \xi_m) = c(\xi_{m+1}, \dots, \xi_n)) \\ 1 & , \text{otherwise} \end{cases}$$

Let $H \in [\kappa]^\kappa$ be a homogeneous set for $\tilde{c}$. Let $n = 2m < \omega$ be given. We must have that there are $s, t \in [H]^m$ satisfying both that $\max(s) < \min(t)$ and $c(s) = c(t)$, as otherwise $\delta \geqslant \kappa$. But then $\tilde{c}(s \cup t) = 0$, meaning $\tilde{c}"[H]^n = \{0\}$ for even numbers $n < \omega$ by homogeneity.

Now take $s, t \in [H]^m$ for any $m < \omega$. Then $s \cup t \in [H]^{2m}$, so $\tilde{c}(s \cup t) = 0$, meaning $c(s) = c(t)$, whence we may conclude that H is homogeneous for c as well. ∎

Proposition 3.1.4. *A cardinal κ is Ramsey iff for every $\delta < \kappa$ and every countable collection of colourings $c_i : [\kappa]^{m_i} \to \delta$ with $m_i < \omega$ there's a set $H \in [\kappa]^\kappa$ such that $|c_i"[H]^{m_i}| \leqslant 1$ for every $i < \omega$.*

Proof. "$\Leftarrow$": Let $c : [\kappa]^{<\omega} \to \delta$ and choose $H \in [\kappa]^\kappa$ for the set $\{c \restriction [\kappa]^n \mid n < \omega\}$.
"$\Rightarrow$": Let $X := \{c_i : [\kappa]^{m_i} \to \delta\}$ be given. Note that given a colouring $c : [\kappa]^m \to \delta$ and $n \geqslant m$ we can always extend c to $\tilde{c} : [\kappa]^n \to \delta$ such that a homogeneous set for $\tilde{c}$ is also a homogeneous set for c. Indeed, just set $\tilde{c}(\{\alpha_1, \dots, \alpha_n\}) := c(\{\alpha_1, \dots, \alpha_m\})$.

Now we can just recursively extend the c_i's such that their domains are disjoint, so that we get an induced colouring $c : [\kappa]^{<\omega} \to \delta$ which then has a homogeneous set as κ is Ramsey. But then every c_i has a homogeneous set as well by the previous statement. ∎

3.2 Kleene-Brouwer ordering

Our argument is going to make crucial use of a well-ordering property implied by the existence of a Ramsey cardinal. The well-ordering property will concern an ordering called the *Kleene-Brouwer ordering*, which we'll introduce here.

Definition 3.2.1. The **Kleene-Brouwer ordering** on ${}^{<\omega}\mathrm{On}$ is given by

$$s <_{\mathrm{KB}} t \Leftrightarrow s \supseteq t \vee s(\iota) < t(\iota),$$

where $\iota := \min\{i < \omega \mid s(i) \neq t(i)\}$. ○

One can (informally) think of the ordering as $s <_{\mathrm{KB}} t$ iff $s^\frown\langle\infty,\infty,\dots\rangle$ is lexicographically less than $t^\frown\langle\infty,\infty,\dots\rangle$, where ∞ is formally greater than all the ordinals.

Lemma 3.2.2. *Let $\alpha \in \mathrm{On}$ and T a tree on α. Then T is well-founded iff T is well-ordered by $<_{KB}$.*

Proof. We'll prove that T is ill-founded iff $(T, <_{\mathrm{KB}})$ is ill-founded.

" $\Rightarrow$ ": Let $\{s_i \in T \mid i < \omega\}$ satisfy $s_{i+1} <_{\mathrm{KB}} s_i$ for every $i < \omega$. For $i < \omega$ set $n_i := \min\{s_j(i) \mid j < \omega\}$. Note that n_i is non-empty for every $i < \omega$ as otherwise $\{s_i \in T \mid i < \omega\} \subseteq {}^{<n}\alpha$ for some $n < \omega$, and $({}^{<n}\alpha, <_{\mathrm{KB}})$ is clearly a well-ordering, which contradicts how we defined the s_i's. Now we have an infinite branch $x \in [T]$ defined as $x(i) := n_i$, which means that T is ill-founded.

" $\Leftarrow$ ": Let $x \in [T]$ be an infinite branch. Then $x \upharpoonright n+1 <_{\mathrm{KB}} x \upharpoonright n$ for every $n < \omega$, so $(T, <_{\mathrm{KB}})$ is ill-founded. ∎

3.3 Analytic determinacy

Definition 3.3.1. **Analytic determinacy** is the statement that every analytic game $G_\omega(A)$ over $\mathcal{N}$ is determined. ○

Lemma 3.3.2. *Let $A \subseteq \mathcal{N}$ be analytic. Then there is a tree on $\omega\times\omega$ such that $x \notin A$ iff $(T_x, <_{KB})$ embeds into $(\omega_1, <)$, where $<_{KB}$ is the Kleene-Brouwer ordering on ${}^{<\omega}\mathrm{On}$ and T_x is the section tree on ω:*

$$T_x := \{p \in {}^{<\omega}\omega \mid (x \upharpoonright \mathrm{len}(p), p) \in T\}.$$

Proof. Since analytic sets are projections of closed sets, there's a closed set $C \subseteq \mathcal{N} \times \mathcal{N}$ such that $A = \pi_1(C)$, where $\pi_1 : \mathcal{N}\times\mathcal{N} \to \mathcal{N}$ is the projection onto the first coordinate. But then there's a tree $T \subseteq {}^{<\omega}(\omega\times\omega)$ such that $[T] = C$. By now we then have that

$$\begin{aligned} x \notin A &\Leftrightarrow \forall y \in C : \pi_1(y) \neq x \\ &\Leftrightarrow T_x \text{ is well-founded} \\ &\Leftrightarrow (T_x, <_{\text{KB}}) \text{ is well-ordered (Lemma 3.2.2)} \\ &\Leftrightarrow (T_x, <_{\text{KB}}) \text{ embeds into } (\omega_1, <), \end{aligned}$$

so we're done. ■

Now for the main result of this section, that it's relatively consistent to assume that analytic determinacy is true:

Theorem 3.3.3 (Martin). *If there exists a Ramsey cardinal then analytic determinacy holds.*

Proof. Let $\{s_i \mid i < \omega\}$ be the enumeration of ${}^{<\omega}\omega$. Let $A \subseteq \mathcal{N}$ be analytic and let T be given by Lemma 3.3.2, meaning that

$$x \notin A \Leftrightarrow (T_x, <_{\text{KB}}) \text{ embeds into } (\omega_1, <). \tag{1}$$

Let κ be a Ramsey cardinal, $G_\omega(A)$ be given and define the game $G^\star$ as

$$\begin{array}{lccccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ \text{II} & & (x_1, \eta_0) & & (x_3, \eta_1) & & (x_5, \eta_2) & & \cdots \end{array}$$

where $x_i \in \omega$ and $\eta_i \in \kappa$. Player II wins this game iff

(i) $(\forall s_i \notin T_x)(\eta_i = 0)$ and

(ii) $(\forall s_i, s_j \in T_x)(s_i <_{\text{KB}} s_j \Leftrightarrow \eta_i < \eta_j)$.

By (1), this means that if player II wins in $G^\star$ he also wins in $G_\omega(A)$, since the win in $G^\star$ grants him with an embedding $f : (T_x, <_{\text{KB}}) \to (\kappa, <)$ given by $f(s_i) := \eta_i$, which induces an embedding $(T_x, <_{\text{KB}}) \to (\omega_1, <)$ as T_x is countable.

Note that since $G^\star$ is an open game for player I since the payoff set defined by the negation of (i) and (ii) above are open criteria. Hence $G^\star$ is determined by the Gale-Stewart Theorem 2.1.3.

All that remains to show is therefore that if player I has a winning strategy $\sigma^\star$ in $G^\star$ then player I also has a winning strategy in $G_\omega(A)$. For any $s \in {}^{2n}\omega$ define the set

$$D_s := \{s_i \in {}^{<\omega}\omega \mid i < n \wedge (s \upharpoonright \mathrm{len}(s_i), s_i) \in T\}$$

and note that $s \subseteq t \Rightarrow D_s \subseteq D_t$. Furthermore we see that $T_x = \bigcup\{D_s \mid s \subseteq x\}$. Let now $p \in {}^{2n}\omega$ be given and set $m := |D_p|$. Then for any $Q \in [\kappa]^m$ there's a *unique* function $n \to \kappa$ defined by $i \mapsto \xi_i^{p,Q}$ satisfying

- $(\forall s_i \notin D_p)(\xi_i^{p,Q} = 0)$;
- $(\forall s_i \in D_p)(\xi_i^{p,Q} \in Q)$;
- $(\forall s_i, s_j \in D_p)(s_i <_{\mathrm{KB}} s_j \Leftrightarrow \xi_i^{p,Q} < \xi_j^{p,Q})$.

Existence and uniqueness of such a function is clear, making us able to define a colouring $c_p : [\kappa]^m \to \omega$ for every $p = \langle p_0, \dots, p_{2n-1}\rangle \in {}^{2n}\omega$ as

$$c_p(Q) := \sigma^\star(\langle p_0, (p_1, \xi_0^{p,Q}), p_2, (p_3, \xi_1^{p,Q}), \dots, p_{2n-2}, (p_{2n-1}, \xi_{n-1}^{p,Q})\rangle).$$

Now use that κ is Ramsey and Proposition 3.1.4 to find a homogeneous set $H \subseteq \kappa$ with $|H| = \aleph_1$ for the countably many c_p with $\mathrm{len}(p)$ even – i.e. satisfying that $|c_p``[H]^n| = 1$ for $p \in {}^{2n}\omega$. Then define a strategy σ for player I in $G_\omega(A)$ as

$$\sigma(p) := c_p(Q),$$

for any, hence all, $Q \in [H]^n$, where $p \in {}^{2n}\omega$.

To see that σ is winning, assume towards a contradiction that it's not, meaning that there's a play $x \in \mathcal{N}$ such that $x \notin A$ even though x followed σ. By (1) this means that we have an embedding $\tilde{\eta} : (T_x, <_{\mathrm{KB}}) \to (\omega_1, <)$ and since $|H| = \aleph_1$ we also have an embedding

$$\eta : (T_x, <_{\mathrm{KB}}) \to (H, <),$$

which can be extended to entire ${}^{<\omega}\omega$ by setting $\eta(t) := 0$ for $t \notin T_x$. Setting $\bigcup_{n<\omega} {}^{2n}\omega = \{u_i \mid i < \omega\}$ be an enumeration, we see that

$$\begin{array}{lccccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ \text{II} & & (x_1, \eta(u_0)) & & (x_3, \eta(u_1)) & & (x_5, \eta(u_2)) & & \cdots \end{array}$$

is a play in $G^\star$ consistent with $\sigma^\star$ as every $\eta(u_i) \in H$, so $x \in A$, $\lightning$. Hence σ is winning and $G_\omega(A)$ is determined. ■

Note that we've only shown that analytic determinacy is relatively consistent with the existence of a Ramsey cardinal, so Ramsey is an "upper bound" for consistency. It turns out that the exact consistency strength of analytic determinacy is the existence of $x^\sharp$ for every real x – see [Kan09, Theorem 31.2 & 31.5].

4 Blackwell determinacy

We now turn to a class of completely different games, namely games involving "chance". These games can be thought of as generalisations of rock-paper-scissors (without draws), where two players take turns simultaneously.

This means that strategies aren't as straightforward as with the perfect information games, since the active player doesn't know what the other player will play in the active round. To model this scenario we'll introduce a new notion of strategies, based on *probability*.

4.1 Blackwell games

Say that a tree $T \subseteq {}^{\omega}X$ on any set X is **finitely branching** if there are only finitely many legal moves at each position, i.e. that given any $p \in T$ there is a finite subset $X_p \subseteq X$ such that $p^\frown\langle x\rangle \in T$ iff $x \in X_p$.

Let X, Y be sets and $T \subseteq {}^{\omega}(X \times Y)$ a pruned non-empty finitely branching tree $T \subseteq {}^{\omega}(X \times Y)$. Then we can define a **Blackwell game** $\Gamma_{X,Y}(T)$ between two players I and II, who *simultaneously* play elements of X and Y, respectively:

$$\begin{array}{lllll} \mathrm{I} & x_0 & x_1 & x_2 & \cdots \\ \mathrm{II} & y_0 & y_1 & y_2 & \cdots \end{array}$$

For a partial play $p \in T$, write $X_p \subseteq X$ and $Y_p \subseteq Y$ for the finite sets of legal moves at p for player I and II, respectively. A **partial play** in a Blackwell game is then a finite sequence of ordered pairs $\langle(x_0, y_0), \dots, (x_n, y_n)\rangle$, and we require that all partial plays lie in T, the tree of legal moves. This results in a **play** $\langle(x_i, y_i)\rangle \in {}^{\omega}(X \times Y)$.

A **strategy** for player I in $\Gamma_{X,Y}(T)$ is a function $\sigma : T \to [0,1]^X$ which satisfies that $\sum_{x \in X_p} \sigma_p(x) = 1$ and $\sigma_p(x) = 0$ for every $x \notin X_p$, i.e. a function which assigns to each position a probability distribution on the legal moves at that position. Strategies for player II are defined analogously. Thus instead of having a definite choice of what player I should do, a strategy merely tells player I what the odds are.

Two strategies σ and τ for players I and II, respectively, give rise to a Borel probability measure $\mu_{\sigma,\tau}$ on $[T]$, defined as follows. We start by defining $\mu_{\sigma,\tau}(N_s)$ for $s \in T$ by recursion on $\mathrm{len}(s)$. For $\mathrm{len}(s) = 0$ set $\mu_{\sigma,\tau}(N_s) = 1$ and if s is a one-point extension of p then set

$$\mu_{\sigma,\tau}(N_s) := \sigma(\pi_1(s(\mathrm{len}(s) - 1))) \cdot \tau(\pi_2(s(\mathrm{len}(s) - 1))) \cdot \mu_{\sigma,\tau}(N_p),$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections. Then since every open set in $[T]$ is a countable disjoint union of basic opens by Proposition A.1.7, we set $\mu_{\sigma,\tau}(\coprod_{s_i} N_{s_i}) := \sum_{s_i} \mu_{\sigma,\tau}(N_{s_i})$. Finally, extend to all Borel sets by setting $\mu_{\sigma,\tau}(\neg A) := 1 - \mu_{\sigma,\tau}(A)$ and $\mu_{\sigma,\tau}(\coprod_{i<\omega} A_i) := \sum_{i<\omega} \mu_{\sigma,\tau}(A_i)$.

To be able to determine whether or not a strategy is winning, we define a **payoff function** $f : [T] \to B$ where $B \subseteq \mathbb{R}$ is some bounded subset of the reals. Denote the associated game by $\Gamma := \Gamma_{X,Y}(T, f)$. If f turns out to be $\mu_{\sigma,\tau}$-measurable then define the **expectation**

$$E_{\sigma,\tau}(f) := \int f d\mu_{\sigma,\tau},$$

which indicates what the net outcome of the game for player I will be. Thus player I tries to maximize $E_{\sigma,\tau}(f)$ and player II tries to minimize it. To be able to deal with non-measurable payoff functions as well, we define

$$E^-_{\sigma,\tau}(\Gamma) := \sup\{E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable} \wedge g \leqslant f\}$$
$$E^+_{\sigma,\tau}(\Gamma) := \inf\{E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable} \wedge g \geqslant f\},$$

which is then best approximations of the outcome for player II and I, respectively. Note that $E^-_{\sigma,\tau}(\Gamma) = E_{\sigma,\tau}(\Gamma) = E^+_{\sigma,\tau}(\Gamma)$ if f is $\mu_{\sigma,\tau}$-measurable. Now, for strategies σ and τ in Γ for player I and II respectively, define

$$\mathrm{val}_\sigma(\Gamma) := \inf\{E^-_{\sigma,\tau}(\Gamma) \mid \tau \text{ is a strategy for player II}\}$$
$$\mathrm{val}_\tau(\Gamma) := \sup\{E^+_{\sigma,\tau}(\Gamma) \mid \sigma \text{ is a strategy for player I}\},$$

where val_σ represents what the best player II can do if player I follows σ, and val_τ represents the best player I can do if player II follows τ. Thus the value of a strategy is what the worst outcome is when following the strategy. Finally we set

$$\mathrm{val}_\downarrow(\Gamma) := \sup\{\mathrm{val}_\sigma(\Gamma) \mid \sigma \text{ is a strategy for player I}\}$$
$$\mathrm{val}^\uparrow(\Gamma) := \inf\{\mathrm{val}_\tau(\Gamma) \mid \tau \text{ is a strategy for player II}\}.$$

These values then represent what the best *consistent* outcome is for the two players. If it is the case that $\mathrm{val}_\downarrow(\Gamma) = \mathrm{val}^\uparrow(\Gamma)$ then we say that the game Γ is **determined**. If this is the case, we define the **value of the game** Γ to be

$$\mathrm{val}(\Gamma) := \mathrm{val}_\downarrow(\Gamma) = \mathrm{val}^\uparrow(\Gamma).$$

4.2 Blackwell determinacy

Our goal in this section is to show that determinacy of certain perfect information games implies the determinacy of all Blackwell games. Thus let us throughout the section fix a Blackwell game $\Gamma := \Gamma_{X,Y}(T, f)$.

Since we're only interested in whether or not the game is determined, we have no interest in the scale of the value and we can thus without loss of generality that $\mathrm{ran}\, f \subseteq [0, 1]$. The crucial theorem that we're going to use extensively is the famous Minimax Theorem by von Neumann, assessing that one-step Blackwell games are determined:

Theorem 4.2.1 (von Neumann's Minimax Theorem). *Any Blackwell one-step game Γ is determined in a strong sense: there is a strategy σ for one of the players such that* $\mathrm{val}(\Gamma) = \mathrm{val}_\sigma(\Gamma)$.

Proof. See [vN28]. ■

For brewity, set $\mathcal{E}_T(p)$ to be the set of **one-point extensions** of $p \in T$; i.e. the set of $q \in T$ such that $q = p ^\frown \langle (x, y) \rangle$ for some $x \in X_p$ and $y \in Y_p$.

The perfect information games that we're going to work with will be indexed by $v \in (0, 1]$ and denoted G_v. The game G_v is played as follows:

$$\begin{array}{llllllll} \mathrm{I} & h_0 & & h_1 & & h_2 & & \cdots \\ \mathrm{II} & & p_1 & & p_2 & & p_3 & & \cdots \end{array}$$

Here $p_i \in T$ and $h_i : \mathcal{E}_T(p_i) \to [0, 1] \cap \mathbb{Q}$. The rules are as follows. First of all, $\mathrm{len}(p_i) = i$ and $p_i \subseteq p_{i+1}$ for all $i < \omega$, where we set $p_0 := \langle\rangle$. For $i < \omega$ define the one-step Blackwell game Δ_i where both players simultaneously play elements $x \in X_{p_i}$, $y \in Y_{p_i}$:

$$\begin{array}{ll} \mathrm{I} & x \\ \mathrm{II} & y \end{array}$$

Hence the h_i's can be seen as payoff functions for the Δ_i games via a suitable coding. Now, letting $v_0 := v$ and $v_{i+1} := h_i(p_{i+1})$ for $i < \omega$, we require of p_{i+1} that $v_{i+1} > 0$ and of h_i that $\mathrm{val}(\Delta_i(h_i)) \geqslant v_i$.

Now define $\pi : \tilde{T} \to T$ as taking each partial play $s \in \tilde{T}$ in G_v to the union of all the moves made by player II in arriving at s; hence $\pi(s) = \varnothing$ if $\mathrm{len}(s) \leqslant 1$ and otherwise $\pi(s)$ is the last move made by player II. Define the extension $\tilde{\pi} : [\tilde{T}] \to [T]$ as $\tilde{\pi}(x) := \bigcup_{i<\omega} \pi(x \upharpoonright i)$. Now define the payoff set $A \subseteq \mathcal{N}$ as the set containing all the encoded $x \in [\tilde{T}]$ satisfying that

$$\limsup_{i<\omega} v_i \leqslant f(\pi(x)).$$

This finishes the definition of the game $G_v(A)$.

Proposition 4.2.2. *$G_v(A)$ is equivalent to a perfect information game $G_\omega(\tilde{A})$ for every $v \in (0, 1]$.*

Proof. First of all, it's clear that all the h_i's and p_i's can be encoded as natural numbers since T is finitely branching – note here that it's important that the codomains of the h_i's are countable. We thus just need to argue that the tree $\tilde{T} \subseteq \mathcal{N}$ of legal moves is pruned.

Player I can always just play $h_i :\equiv 1$, which trivially satisfies the requirements for h_i. We show that player II always has a possible legal move by induction on $i < \omega$. Firstly $\mathrm{val}(\Delta_0(h_0)) \geqslant v > 0$ requires that h_0 is not identically zero. Let thus p_1 be such that $h_0(p_1) \stackrel{\text{def}}{=} v_1 > 0$. Now assuming that h_{i+1} has been given, repeat the procedure to get some $v_{i+1} > 0$. ■

Now assume $\tilde{\sigma}$ is a winning strategy for player I in G_v. We'll simultaneously define

(i) A strategy σ for player I in Γ;
(ii) The notion of an *acceptable* position in Γ;
(iii) A monotone function $\psi : \mathrm{acc}(T) \to \tilde{T}$, where $\mathrm{acc}(T)$ is the set of acceptable positions in T, such that $\mathrm{len}(\psi(p)) = 2\,\mathrm{len}(p) + 1$, $\psi(p)$ is consistent with $\tilde{\sigma}$ and $\pi(\psi(p)) = p$.

This will then result in a function $\tilde{\psi} : [\mathrm{acc}(T)] \to [\tilde{T}]$ given by $\tilde{\psi}(x) := \bigcup_{p \subseteq x} \psi(p)$ satisfying that $\tilde{\psi}(x)$ is consistent with $\tilde{\sigma}$ and $\pi(\psi(x)) = x$.

First of all, we define every extension of an unacceptable position to be unacceptable, and given any unacceptable position p we define $\sigma_p : X \to [0,1]$ arbitrarily. Furthermore $\langle\rangle \in \mathrm{acc}(T)$ and we set $\psi(\langle\rangle) = \langle h_0 \rangle$, where $h_0 = \tilde{\sigma}(\langle\rangle)$.

Now assume that $p \in \mathrm{acc}(T)$ is of length $i > 0$ and we've defined $\psi(p) = \langle h_0, \dots, p_i, h_i \rangle$ which is both consistent with $\tilde{\sigma}$ and satisfies $\pi(\psi(p)) = p$. Then we define $q \in \mathcal{E}_T(p)$ to be acceptable iff $h_i(q) > 0$.

Because $\mathrm{val}(\Delta_i(h_i)) \geqslant v_i$, we know by von Neumann's Theorem 4.2.1 that there exists a strategy for player I in Δ_i whose value in $\Delta_i(h_i)$ is $\geqslant v_i$; set $\sigma_p : X \to [0,1]$ to be the probability distribution given by that strategy. Given any $q \in \mathcal{E}_T(p)$ set $\psi(q) := \psi(p)\,^\frown\langle q, h_{i+1} \rangle$ with h_{i+1} given by $\tilde{\sigma}$. This finishes the definition of (i)-(iii).

Now, to define a payoff function g, define first h_i^p for acceptable $p \in T$ and $0 \leqslant i \leqslant \mathrm{len}(p)$ to be the moves made by player I in reaching the position $\psi(p) \in \tilde{T}$. Set $v_0^p := v$ and $v_{i+1}^p := h_i^p(p \restriction i+1)$ for $i < \mathrm{len}(p)$. Then define $g : [T] \to [0,1]$ as

$$g(x) := \begin{cases} \limsup_i v_i^{x \restriction i} & , x \in [\mathrm{acc}(T)] \\ 0 & , x \notin [\mathrm{acc}(T)] \end{cases}$$

Proposition 4.2.3. *g is Borel measurable.*

Proof. It suffices to show that the functions $\alpha_i : [\mathrm{acc}(T)] \to [0,1]$ given by $\alpha_i(x) := v_i^{x\restriction i}$ are Borel measurable for every $i < \omega$, since $\limsup$ of a sequence of Borel measurable functions is Borel measurable. α_0 is constant, so clearly Borel measurable. Let $a, b \in [0,1]$ with $a < b$. Then

$$\begin{aligned} x \in \alpha_{i+1}^{-1}((a,b)) &\Leftrightarrow h_i^{x\restriction i+1}(x \restriction i+1) \in (a,b) \\ &\Leftrightarrow x \restriction i+1 \in (h_i^{x\restriction i+1})^{-1}((a,b)), \end{aligned}$$

so $\alpha_{i+1}^{-1}((a,b)) = \bigcup\{N_{x\restriction i+1} \mid x \restriction i+1 \in (h_i^{x\restriction i+1})^{-1}((a,b))\}$, which is open and thus Borel. ■

Lemma 4.2.4. *Given any strategy τ for player II in Γ, $E_{\sigma,\tau}(\Gamma(g)) \geqslant v$.*

Proof. Let τ be a strategy for player II in Γ and write $\mu := \mu_{\sigma,\tau}$. Assume for a contradiction that

$$E_{\sigma,\tau}(g) \overset{\text{def}}{=} \int g \, d\mu < v. \tag{1}$$

Let $\varepsilon > 0$ be given such that $\int g\, d\mu < v-\varepsilon$, which means that $\int 1-g\, d\mu > 1-v+\varepsilon$. Now by Lusin's Theorem A.5.5, there's a closed set $C \subseteq [T]$ such that $g \restriction C$ is continuous and $\int_C 1-g\, d\mu > 1-v+\varepsilon$.

Claim 4.2.4.1. There exists a play $x \in [\mathrm{acc}(T)]$ such that, for all $i < \omega$,

$$\int_{C\cap N_{x\restriction i}} 1-g\, d\mu > (1-v_i^{x\restriction i}+\varepsilon)\mu(N_{x\restriction i}). \tag{2}$$

Proof of claim. We define x by recursion on $i < \omega$. For $i = 0$, (2) says that $\int_C 1-g\, d\mu > 1-v+\varepsilon$, which was the defining property of C. Assume now that $x \restriction i$ is acceptable such that (2) holds, and assume towards a contradiction that

$$\int_{C \cap N_q} 1 - g \, d\mu \leqslant (1 - h_i^{x \restriction i}(q) + \varepsilon)\mu(N_q) \tag{3}$$

holds for every $q \in \mathcal{E}_T(x \restriction i)$. Note that (3) holds for all unacceptable $q \in \mathcal{E}_T(x \restriction i)$ as well, since then $h_i^{x \restriction i}(q) = 0$ by definition of being unacceptable and thus

$$\int_{C \cap N_q} 1 - g \, d\mu \leqslant \int_{N_q} 1 \, d\mu = \mu(N_q) < (1 + \varepsilon)\mu(N_q).$$

Hence we have that

$$\begin{aligned}
\int_{C \cap N_{x \restriction i}} 1 - g \, d\mu &= \sum_q \int_{C \cap N_q} 1 - g \, d\mu \\
&\leqslant \sum_q (1 - h_i^{x \restriction i}(q) + \varepsilon)\mu(N_q) \\
&= \mu(N_{x \restriction i})(1 + \varepsilon) - \sum_q h_i^{x \restriction i}(q)\mu(N_q) \\
&= \mu(N_{x \restriction i})(1 + \varepsilon - E_{\sigma,\tau}(h_i^{x \restriction i})) \\
&\leqslant (1 - v_i^{x \restriction i} + \varepsilon)\mu(N_{x \restriction i}),
\end{aligned} \tag{4}$$

where the last inequality holds because we chose σ such that $\mathrm{val}_\sigma(h_i^{x \restriction i}) \geqslant v_i^{x \restriction i}$ in the game $\Delta_i^{x \restriction i}$ (which is completely analogous to the game Δ_i), meaning that $E_{\sigma,\tau}(h_i^{x \restriction i}) \geqslant v_i^{x \restriction i}$ holds for every strategy τ for player II.

But now we see that (4) contradicts our induction hypothesis that $x \restriction i$ satisfies (2), so we conclude that there is a $q \in \mathcal{E}_T(x \restriction i)$ such that (3) fails, i.e. that

$$\int_{C \cap N_q} 1 - g \, d\mu > (1 - h_i^{x \restriction i}(q) + \varepsilon)\mu(N_q)$$

holds. Thus since we showed that $h_i^{x \restriction i}(q) \geqslant v_i^{x \restriction i}$, (2) holds for $i + 1$ and the claim is proven. $\diamondsuit$

Now observe that for any $i < \omega$ there's a $y_i \in C \cap N_{x \restriction i}$ such that

$$g(y_i) < v_i^{x \restriction i} - \varepsilon \tag{5}$$

holds. Indeed, assuming that (5) fails for every $y_i \in C \cap N_{x \restriction i}$, we would have that

$$\int_{C \cap N_{x \restriction i}} 1 - g \, d\mu \leqslant (1 - v_i^{x \restriction i} + \varepsilon)\mu(N_{x \restriction i}),$$

contradicting the above claim.

As $\lim_i y_i = x$, we have that $x \in C$ as C is closed, and thus $g(x) = \lim_i g(y_i)$ by continuity of $g \restriction C$. Use continuity to pick $j < \omega$ such that $|g(x) - g(y_i)| < \varepsilon/2$ for every $i \geqslant j$. This means that $g(x) < g(y_i) + \varepsilon/2 < v_i^{x \restriction i} - \varepsilon/2$, so

$$g(x) \leqslant \limsup_i v_i^{x \restriction i} - \varepsilon/2 = g(x) - \varepsilon/2,$$

a contradiction. This means that (1) is false, so the lemma is proven. ■

Lemma 4.2.5. $\mathrm{val}_\sigma(\Gamma(f)) \geqslant v$.

Proof. By the above lemma we see that $\mathrm{val}_\sigma(\Gamma(g)) \geqslant v$. But since $\tilde{\sigma}$ was winning, we have that

$$g(x) = \limsup_{i<\omega} v_i^{x \restriction i} \leqslant f(\pi(\psi(x))) = f(x)$$

for every $x \in [\mathrm{acc}(T)]$, by definition of the $v_i^{x \restriction i}$'s and the rules of the game G_v. Hence $\mathrm{val}_\sigma(\Gamma(f)) \geqslant \mathrm{val}_\sigma(\Gamma(g)) \geqslant v$. ■

Dropping the assumption on $\tilde{\sigma}$, we've shown the following.

Lemma 4.2.6. *If player I has a winning strategy in G_v then* $\mathrm{val}_\downarrow(\Gamma(f)) \geqslant v$. ■

We now shift the attention to player II. Assume thus that $\tilde{\tau}$ is a winning strategy for player II in G_v. Repeating the procedure as before, we'll simultaneously define

(i) A strategy τ for player II in Γ;

(ii) The notion of an acceptable position in Γ;
(iii) For each acceptable position p, a function $u_p : \mathcal{E}_T(p) \to [0,1]$;
(iv) A monotone function $\psi : \mathrm{acc}(T) \to \tilde{T}$ such that $\mathrm{len}(\psi(p)) = 2\,\mathrm{len}(p)$, $\psi(p)$ is consistent with $\tilde{\tau}$ and $\pi(\psi(p)) = p$.

Just as before, this will result in a function $\tilde{\psi} : [\mathrm{acc}(T)] \to [\tilde{T}]$ given by $\tilde{\psi}(x) := \bigcup_{p \subseteq x} \psi(p)$ such that $\tilde{\psi}(x)$ is consistent with $\tilde{\tau}$ and $\pi(\psi(x)) = x$.

Every extension of an unacceptable position is unacceptable, and given any unacceptable position p we define $\tau_p : X \to [0,1]$ arbitrarily. Furthermore $\langle\rangle \in \mathrm{acc}(T)$ and we set $\psi(\langle\rangle) := \langle\rangle$.

Now suppose that $p \in \mathrm{acc}(T)$ is of length $i > 0$ and we've defined $\psi(p) = \langle h_0, \dots, p_i \rangle$ which is consistent with $\tilde{\tau}$ and satisfies $\pi(\psi(p)) = p$ (i.e. that $p_i = p$ by definition of π). Then $q \in \mathcal{E}_T(p)$ is to be acceptable iff there is a legal move $h \in \tilde{T}$ for player I at $\psi(p)$ such that $\tilde{\tau}(\psi(p)^\frown\langle h \rangle) = q$. As for the definition of u_p, set $u_p(q) := 0$ for $q \notin \mathrm{acc}(T)$ and

$$u_p(q) := \inf\{h(q) \mid \ulcorner h \text{ is legal in } G_v \text{ at } \psi(p)\urcorner \wedge \tilde{\tau}(\psi(p)^\frown\langle h \rangle) = q\}$$

for $q \in \mathrm{acc}(T)$.

Lemma 4.2.7. $\mathrm{val}(\Delta_i(u_p)) \leqslant v_i$.

Proof. Assume for a contradiction that $\mathrm{val}(\Delta_i(u_p)) > v_i$ and let $\varepsilon > 0$ be such that $\mathrm{val}(\Delta_i(u_p)) \geqslant v_i + \varepsilon$. Define a function $h : T \to [0,1]$ by

$$h(q) := \begin{cases} u_p(q) - \varepsilon & , u_p(q) > \varepsilon \\ 0 & , u_p(q) \leqslant \varepsilon \end{cases}$$

Then h is a legal move for player I at the position $\psi(p)$. Now set $\bar{q} := \tilde{\tau}(\psi(p)^\frown\langle h \rangle)$. If $u_p(\bar{q}) \leqslant \varepsilon$ then $h(\bar{q}) = 0$, meaning q is *not* a legal move, $\lightning$. Hence $u_p(\bar{q}) > \varepsilon$, but then $h(\bar{q}) < u_p(\bar{q})$ and $u_p(\bar{q})$ was the least such, $\lightning$. We conclude that $\mathrm{val}(\Delta_i(u_p)) \leqslant v_i$. ∎

The claim, along with von Neumann's Theorem 4.2.1, then gives us a strategy for player II in Δ_i whose value in $\Delta_i(u_p)$ is $\leqslant v_i$; set $\tau_p : X \to [0,1]$ to be the probability distribution given by that strategy.

Lastly we define ψ. Fix some $\delta > 0$. Let $q \in \mathcal{E}_T(p)$ be acceptable and h_i a legal move for player I at $\psi(p)$ so that $h_i(q) \leqslant u_p(q) + \delta/2^{i+1}$ and $\tilde{\tau}(\psi(p)\,^\frown\langle h_i\rangle) = q$. Then set $\psi(q) := \psi(p)\,^\frown\langle h_i, q\rangle$. This finishes the definition of (i)-(iv).

Again we'd like to define a payoff function g, so analogously we define h_i^p for $p \in \mathrm{acc}(T)$ and $0 \leqslant i < \mathrm{len}(p)$ as the moves made by player I in reaching the position $\psi(p)$. Let $v_0^p := v$ and $v_{i+1}^p := h_i^p(p \restriction i+1)$ for $i < \mathrm{len}(p)$. Then define

$$g(x) := \begin{cases} \limsup_i v_i^{x \restriction i} & , x \in [\mathrm{acc}(T)] \\ 1 & , x \notin [\mathrm{acc}(T)]. \end{cases}$$

Just as before, we note that g is Borel measurable.

Lemma 4.2.8. *For any strategy σ for player I in Γ, $E_{\sigma,\tau}(\Gamma(g)) \leqslant v + \delta$.*

Proof. Let σ be a strategy for player I in Γ and set $\mu := \mu_{\sigma,\tau}$. Assume for a contradiction that

$$E_{\sigma,\tau}(\Gamma(g)) \stackrel{\text{def}}{=} \int g \, d\mu > v + \delta \tag{1}$$

and let $\varepsilon > 0$ be given such that $\int g \, d\mu > v + \delta + \varepsilon$. Let $C \subseteq [T]$ be a closed set such that $g \restriction C$ is continuous and $\int_C g \, d\mu > v + \delta + \varepsilon$, given by Lusin's Theorem A.5.5.

Claim 4.2.8.1. There exists a play $x \in [\mathrm{acc}(T)]$ such that, for all $i < \omega$,

$$\int_{C \cap N_{x \restriction i}} g \, d\mu > (v_i^{x \restriction i} + \delta/2^i + \varepsilon)\mu(N_{x \restriction i}). \tag{2}$$

Proof of claim. As $x \restriction 0 := \langle\rangle$, (2) says that $\int_C g \, d\mu > v + \delta + \varepsilon$, which was the defining property of C. Now assume that $x \restriction i$ has been defined such that $x \restriction i$ is acceptable and (2) holds. Assume for a contradiction that

$$\int_{C \cap N_q} g \, d\mu \leqslant (h_i^q(q) + \delta/2^{i+1} + \varepsilon)\mu(N_q) \tag{3}$$

for every $q \in \mathcal{E}_T(x \upharpoonright i)$. But then

$$\begin{aligned}
\int_{C\cap N_{x\upharpoonright i}} g\, d\mu &= \sum_q \int_{C\cap N_q} g\, d\mu \\
&\leqslant \sum_{q\in \mathrm{acc}(T)} (h_i^q(q) + \delta/2^{i+1} + \varepsilon)\mu(N_q) + \sum_{q\notin \mathrm{acc}(T)} \mu(C\cap N_q) \\
&\leqslant \sum_q (u_{x\upharpoonright i}(q) + \delta/2^{i+1} + \delta/2^{i+1} + \varepsilon)\mu(N_q) \\
&= \mu(N_{x\upharpoonright i})(\delta/2^i + \varepsilon) + \sum_q u_{x\upharpoonright i}(q)\mu(N_q) \\
&= \mu(N_{x\upharpoonright i})(\delta/2^i + \varepsilon + E_{\sigma,\tau}(u_{x\upharpoonright i})) \\
&\leqslant (v_i^{x\upharpoonright i} + \delta/2^i + \varepsilon)\mu(N_{x\upharpoonright i}),
\end{aligned} \tag{4}$$

where the second inequality is because $h_i^q(q) \leqslant u_{x\upharpoonright i}(q) + \delta/2^{i+1}$ for acceptable q by definition of h_i^q, and $u_{x\upharpoonright i}(q) = 1$ for q unacceptable. The last inequality is due to the fact that τ was chosen such that in $\Delta_i^{x\upharpoonright i}(u_{x\upharpoonright i})$, $\mathrm{val}_\tau(u_{x\upharpoonright i}) \leqslant v_i^{x\upharpoonright i}$, meaning that for any strategy σ for player I in $\Delta_i^{x\upharpoonright i}$ we have that $E_{\sigma,\tau}(u_{x\upharpoonright i}) \leqslant v_i^{x\upharpoonright i}$. But we see that (4) contradicts (3), so the claim is shown. ◇

Now we note that given any $i < \omega$ there's a $y_i \in C \cap N_{x\upharpoonright i}$ such that

$$g(y_i) > v_i^{x\upharpoonright i} + \delta/2^i + \varepsilon. \tag{5}$$

Indeed, assume that (5) fails for every $y \in C \cap N_{x\upharpoonright i}$. Then $\int_{C\cap N_{x\upharpoonright i}} g\, d\mu \leqslant (v_i^{x\upharpoonright i} + \delta/2^i + \varepsilon)\mu(N_{x\upharpoonright i})$, contradicting (2).

As $x = \lim_i y_i$, $x \in C$ as C is closed, so $g(x) = \lim_i g(y_i)$ by continuity of $g \upharpoonright C$. Let $j < \omega$ be such that $|g(x) - g(y_i)| < \varepsilon/2$ for every $i \geqslant j$. Hence for every $i \geqslant j$ we also have

$$g(x) > g(y_i) - \varepsilon/2 > v_i^{x\upharpoonright i} + \delta/2^i + \varepsilon/2,$$

concluding that $g(x) \geqslant \limsup_i v_i^{x\upharpoonright i} + \varepsilon/2 = g(x) + \varepsilon/2$, ↯. Hence (1) is false, so the lemma is proven. ■

LEMMA 4.2.9. $\mathrm{val}_\tau(\Gamma(f)) \leqslant v + \delta$.

PROOF. By the above lemma, $\mathrm{val}_\tau(\Gamma(g)) \leqslant v + \delta$. Since $\tilde{\tau}$ was winning, we get that

$$g(x) = \limsup_{i<\omega} v_i^{x \restriction i} > f(\pi(\psi(x))) = f(x)$$

for every $x \in [\mathrm{acc}(T)]$, by definition of the $v_i^{x \restriction i}$'s and the rules of the game G_v. Hence $\mathrm{val}_\tau(\Gamma(f)) \leqslant \mathrm{val}_\tau(\Gamma(g)) \leqslant v + \delta$. ■

Dropping the assumption on $\tilde{\tau}$, we've shown the following.

LEMMA 4.2.10. *If player II has a winning strategy in G_v then* $\mathrm{val}^{\uparrow}(\Gamma(f)) \leqslant v$. ■

All the previous lemmas culminate in the following theorem.

THEOREM 4.2.11. *If G_v is determined for every $v \in (0,1]$, then $\Gamma(f)$ is determined.*

PROOF. Assume that all the G_v are determined and set $w \in (0,1]$ to be the supremum of the v's such that player I has a winning strategy in G_v. By Lemma 4.2.6, $\mathrm{val}_{\downarrow}(\Gamma(f)) \geqslant w$. By Lemma 4.2.10, $\mathrm{val}^{\uparrow}(\Gamma(f)) \leqslant w$. Hence $\mathrm{val}(\Gamma(f)) = w$. ■

DEFINITION 4.2.12. The **Axiom of Blackwell Determinacy**, Bl-AD, is the statement that every Blackwell game $\Gamma_{X,Y}(T, f)$ is determined. ○

THEOREM 4.2.13. ZF $\vdash$ *AD* $\to$ *Bl-AD.*

PROOF. By Proposition 4.2.2, all the G_v's are equivalent to a game of the form $G_\omega(\tilde{T})$, which are determined by AD. Hence by Theorem 4.2.11, all Blackwell games are determined. ■

A | Preliminaries

A.1 Polish spaces and trees

Definition A.1.1. A **Polish space** is a completely metrizable separable space. ○

Proposition A.1.2. *For any discrete space X, ${}^{\omega}X$ is Polish with the product topology.* ■

Thus in particular the **Baire space** $\mathcal{N} := {}^{\omega}\omega$ and the **Cantor space** $\mathcal{C} := {}^{\omega}2$ are both Polish.

Definition A.1.3. Let X be a set. Then a **tree** on X is a subset $T \subseteq {}^{<\omega}X$ which is closed under intial segments – i.e. that for any $t \in T$ and $s \in {}^{<\omega}X$ it holds that $s \subseteq t \Rightarrow s \in T$. ○

Definition A.1.4. A tree T is **pruned** if any branch can be extended, i.e. that for $s \in T$ there exists $t \in T$ such that $s \subsetneq t$. ○

Definition A.1.5. The **body** of a tree T on X, denoted $[T]$, is the set of *infinite branches* of T, i.e. $[T] := \{x \in {}^{\omega}X \mid \forall n < \omega : x \upharpoonright n \in T\}$. ○

We give ${}^{\omega}X$ the product topology, so that the basic open sets become sets of the form $N_s := \{x \in {}^{\omega}X \mid s \subseteq x\}$ for $s \in {}^{<\omega}X$. Given any tree T on X, we endow it with the corresponding subspace topology.

Proposition A.1.6. *Let X be a set and T a pruned tree on X. Then $[T]$ is closed, and any closed set $C \subseteq {}^{\omega}X$ arises in this way; that is, there's a pruned tree T on X such that $C = [T]$.* ■

Proposition A.1.7 (ZF). *Suppose that $s, t \in {}^{<\omega}X$. Then*

(i) *$N_s \cap N_t$ is either $\emptyset$, N_s or N_t;*

(ii) $N_s - N_t$ *is a disjoint union of basic open sets;*

(iii) N_s *is clopen;*

(iv) *Every open set* $U \subseteq {}^{\omega}X$ *is a disjoint union of basic open sets.*

PROOF. See [Kan09, Exercise 0.8] ■

A.2 Borel and analytic sets

DEFINITION A.2.1. Let X be a topological space. Then the **Borel sets** of X, denoted $\mathbb{B}(X)$, is the smallest σ-algebra containing all the open sets in X. ○

We can also characterize the Borel sets more constructively. For X a topological space, define $\mathbf{\Sigma}^0_1(X)$ to be the set of open sets in X and then recursively define $\mathbf{\Pi}^0_\xi := \neg\mathbf{\Sigma}^0_\xi \cup \mathbf{\Sigma}^0_\xi$ and $\mathbf{\Sigma}^0_{\xi+1} := \{\bigcup_{i<\omega} A_i \mid A_i \in \bigcup_{\eta<\xi+1} \mathbf{\Pi}^0_\eta\}$.

PROPOSITION A.2.2. *For any topological space X we have that* $\mathbb{B}(X) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi$. ■

DEFINITION A.2.3. Let X be a Polish space. Then $A \subseteq X$ is **analytic** if it is the continuous image of a Polish space. ○

PROPOSITION A.2.4. *For* X *a Polish space,* $A \subseteq X$ *is analytic iff there exists a closed subset* $C \subseteq X \times \mathcal{N}$ *such that* $\pi_1(C) = A$*, where* $\pi_1 : X \times \mathcal{N} \to X$ *is the first projection.*

PROOF. See [Kec95, Exercise 14.3] ■

THEOREM A.2.5 (Lusin-Souslin). *Let* X *be a Polish space. Then any Borel set* $B \subseteq X$ *is analytic.*

PROOF. See [Kec95, Theorem 13.7]. ■

A.3 Baire property

Definition A.3.1. Let X be a topological space. Then a subset $A \subseteq X$ is **nowhere dense** if $X - \overline{A}$ is dense. A is **meager** if it's a countable union of nowhere dense sets. A is **comeager** if it's the complement of a meager set. ○

Definition A.3.2. Two subspaces $A, B \subseteq X$ are **equal modulo meager sets**, written $A =_* B$, if $A \triangle B$ is meager, where $A \triangle B := A - B \cup B - A$ is the diagonal intersection. ○

Definition A.3.3. For X any topological space, a subset $A \subseteq X$ has the **Baire property** if there exists an open set $U \subseteq \mathbb{R}$ such that $A =_* U$. ○

Theorem A.3.4 (Baire Category Theorem). *Let X be a completely metrizable space. Then any comeager subset $A \subseteq X$ is dense.*

Proof. See [Kec95, Theorem 8.4]. ■

Proposition A.3.5 (AC). *There exists a subset $A \subseteq \mathbb{R}$ not having the Baire property.*

Proof. See [Kec95, Example 8.24]. ■

A.4 Perfect set property

Definition A.4.1. Let X be a topological space. Then $A \subseteq X$ is **perfect** if it has no *isolated points*, i.e. no $a \in A$ such that $\{a\} \subseteq X$ is open. ○

Definition A.4.2. Let X be a Polish space. Then $A \subseteq X$ has the **perfect set property** iff X is either countable or contains a perfect set. ○

Theorem A.4.3. *Let X be a Polish space. Then every analytic subset $A \subseteq X$ has the perfect set property.*

PROOF. See [Kec95, Exercise 14.13]. ■

A.5 Lebesgue measurability

DEFINITION A.5.1. A set $A \subseteq \mathcal{N}$ is **Lebesgue measurable** if there is a Borel set $B \subseteq \mathcal{N}$ such that $A \triangle B$ is a Lebesgue null-set. ○

We denote the Lebesgue measure as λ, both for the Borel measure and the measure on all Lebesgue measurable sets.

LEMMA A.5.2. *Let $A \subseteq \mathcal{N}$ be a Lebesgue measurable set. Then*

(i) For any $\varepsilon > 0$ there is a closed set $C \subseteq \mathcal{N}$ and an open set $U \subseteq \mathcal{N}$ such that $C \subseteq A \subseteq U$ and $\lambda(U - C) < \varepsilon$;

(ii) There is an F_σ set X and a G_δ set Y such that $X \subseteq A \subseteq Y$ and $\lambda(X) = \lambda(A) = \lambda(Y)$.

PROOF. See [Kan09, Lemma 0.9] ■

DEFINITION A.5.3. A measurable space $(X, \mathcal{A})$ is a **standard Borel space** if there's a Polish topology $\mathcal{T}$ on X such that $\mathbb{B}(\mathcal{T}) = \mathcal{A}$. ○

THEOREM A.5.4 (Lusin). *Let X be a standard Borel space. Then every analytic subset $A \subseteq X$ is Lebesgue measurable.*

PROOF. See [Kec95, Theorem 29.7]. ■

THEOREM A.5.5 (Lusin). *Let X be a metrizable space and μ a finite Borel measure on X. Let Y be a second countable topological space and $f : X \to Y$ a μ-measurable function. For every $\varepsilon > 0$ there is a closed set $C \subseteq X$ with $\mu(X - C) < \varepsilon$ such that $f \upharpoonright C$ is continuous.*

PROOF. See [Kec95, Theorem 17.12]. ■

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